

# Geometric approach to stable homotopy groups of spheres. I. The Hopf invariant

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dedicated to the memory of Professor M.M. Postnikov

## Abstract

A geometric approach to the stable homotopy groups of spheres is developed in this paper, based on the Pontryagin-Thom construction. The task of this approach is to obtain an alternative proof of the Hill-Hopkins-Ravenel theorem [H-H-R] on Kervaire invariants in all dimensions, except, possibly, a finite number of dimensions. In the framework of this approach, the Adams theorem on the Hopf invariant is studied, for all dimensions with the exception of 15, 31, 63, 127. The new approach is based on the methods of geometric topology.

## Introduction

Let  $\pi_{n+m}(S^m)$  be the homotopy groups of spheres. Under the condition  $m \geq n+2$  this group is independent of  $m$  and is denoted by  $\Pi_n$ . It is called the stable homotopy group of spheres in dimension  $n$ . The problem of calculating the stable homotopy groups of spheres is one of the main unsolved problems of topology. A development of the Pontryagin-Thom construction leads to various applications having important practical significance: an approach by V.I. Arnol'd to bifurcations of critical points in multiparameter families of functions [V] chapter 3, section 2.2 and Theorem 1, section 2.4, Lemma 4, approximation of maps by embeddings [Me], and gives many unsolved geometrical problems [E2].

For the calculation of elements of the stable homotopy groups of spheres, one frequently studies algebraic invariants which are defined for all dimensions at once (or for some infinite sequence of dimensions). Nevertheless, as a rule these invariants turn out to be trivial, and are nonzero only in exceptional cases, see [M1]. As Prof. Peter Landweber noted: "This a very interesting "philosophy". Are there examples to illustrate this, apart from

the Hopf invariant and the Kervaire invariant? There might be one in N. Minami's paper [M2]."

A basic invariant is the Hopf invariant, which is defined as follows in the framework of stable homotopy theory. The Hopf invariant (also called the stable Hopf invariant), or the Steenrod-Hopf invariant is a homomorphism

$$h : \Pi_{2k-1} \rightarrow \mathbb{Z}/2,$$

for details see [W],[M-T]. The stable Hopf invariant is studied in this paper.

The following theorem was proved by J.F.Adams in [A].

**Theorem.** *The stable Hopf invariant  $h : \Pi_n \rightarrow \mathbb{Z}/2$ ,  $n \equiv 1 \pmod{2}$  is a trivial homomorphism if and only if  $n \neq 1, 3, 7$ .*

**Remark.** The case  $n = 15$  was proved by Toda (cf. [M-T] Ch. 18).

Later Adams and Atiyah offered an alternative approach to the study of the Hopf invariant, based on results of  $K$ -theory and the Bott periodicity theorem, cf. [A-A]. This approach was also extended in subsequent works. A simple proof of the theorem of Adams, close to the proof of Adams and Atiyah, was given by V.M. Buchstaber in [B], section 2.

The definition of the stable Hopf invariant is reformulated in the language of the cobordism groups of immersions of manifolds [E1, K2, K-S1, K-S2, La]. Using the Pontryagin-Thom theorem in the form of Wells on the representation of the stable homotopy groups of infinite dimensional real projective space (which by the Kahn-Priddy theorem surject onto the 2-components of the stable homotopy groups of spheres), we classify the cobordism of immersions of (in general nonorientable) manifolds in codimension 1. The Hopf invariant is expressed as a characteristic number of the manifold of double points of self-intersection of an immersion of a manifold representing the given element of the stable homotopy group. This is explicitly formulated in [E1], Lemma 3.1. This lemma is reformulated in the standard way by means of the Pontryagin-Thom construction for immersions.

The Theorem of Adams admits a simple geometric proof for dimensions  $n \neq 2^\ell - 1$ . In the case  $n \not\equiv 3 \pmod{4}$  a proof, using the elements of the theory of immersions, was given by A. Szűcs in [Sz]. The next case in complexity arises for  $n \neq 2^\ell - 1$ . The proof of Adams' Theorem under this assumption was given by Adem [Adem] using algebraic methods. In this paper the Adem relations on the multiplicative generators of the Steenrod algebra were used. The theorem of Adem was reproved using geometric methods in a joint paper of the author and A. Szűcs [A-Sz].

We assume below that  $n = 2^\ell - 1$ . Define a positive integer  $\sigma = \sigma(\ell)$  by the formula:

$$\sigma = \left\lfloor \frac{\ell}{2} \right\rfloor - 1. \quad (1)$$

In particular, for  $\ell = 8$ ,  $\sigma = 3$ . Denote  $n_s = 2^s - 1$ . Assume that  $s$  is a positive integer, then  $n_s$  is a positive integer.

The following is the main result of Part *I*.

### Main Theorem

Assume that  $\ell \geq 8$ , therefore  $n \geq 255$ . Let  $g : M^{\frac{3n+n\sigma}{4}} \looparrowright \mathbb{R}^n$  be an arbitrary smooth immersion of a closed manifold  $M$ ,  $\dim(M) = \frac{3n+n\sigma}{4}$ , where the normal bundle  $\nu(g)$  to the immersion  $g$  is isomorphic to the Whitney sum of  $(\frac{n-n\sigma}{4})$  copies of a line bundle  $\kappa$  over  $M$ ,  $\nu(g) = (\frac{n-n\sigma}{4})$ . (In particular,  $w_1(M) = 0$ , where  $w_1$  is the first Stiefel-Whitney class, because the codimension of the immersion  $g$  is even and  $w_1(M) = (\frac{n-n\sigma}{4})w_1(\kappa) = 0$ ), in general  $M$  is nonconnected.) Then the equation  $\langle w_1(\kappa)^{\dim(M)}; [M] \rangle = 0$  is valid.

The Main Theorem is deduced from Theorem 12. Theorem 12 is deduced from Propositions 23, 30; these propositions follow from Lemmas 28 and 28. The proofs of these lemmas are given in part *III* [A3].

The proof of the Main Theorem is based on the principle of geometric control due to M.Hirsh, see Proposition 26. This proposition permits one to find within a cobordism class of immersions an immersion with additional properties of self-intersection manifold (see Propositions 23, 30). In this case we say that the immersion admits a cyclic or quaternionic structure (see Definitions ??, 19).

We can deduce the following from the Main Theorem by standard arguments.

### Main Corollary

Let  $g : M^{n-1} \looparrowright \mathbb{R}^n$  be an arbitrary smooth immersion of the closed manifold  $M^{n-1}$ , which in general is not assumed to be orientable. Then under the assumption  $n = 2^\ell - 1$ ,  $n \geq 255$  (i.e., for  $\ell \geq 8$ ), the equality  $\langle w_1(M)^{n-1}; [M] \rangle = 0$  is valid.

**Remark.** The equivalence of the preceding assertion and the theorem of Adams (under the restriction  $\ell \geq 4$ ) is proved in [E1,La].

We mention that in topology there are theorems which are close to the formulation of the theorem of Adams. As a rule, these theorems are corollaries of Adams' theorem. Sometimes these theorems can be given alternative proofs by simpler methods. As S.P. Novikov remarks in his survey [N], a theorem of this type is the Bott-Milnor Theorem that the tangent  $n$ -plane bundle to the standard sphere  $S^n$  is trivial if and only if  $n = 1, 3$  or  $7$ . This theorem was first proved in the paper [B-M]. An elegant modification of the known proof was recently given in [F]. One should mention the Baum-Browder Theorem [B-B] about non-immersion of the standard real projective space  $\mathbb{RP}^{2^{\ell-1}+1}$  into  $\mathbb{R}^{2^{\ell-1}}$  for  $\ell \geq 4$ . It would be interesting to discover an elementary geometrical proof of this theorem and to prove the Main Corollary for  $\ell \geq 4$  as a generalization of Baum-Browder Theorem.

We turn our attention to the structure of the paper. In section 1 we recall the main definitions and constructions of the theory of immersions. The results of this section are formally new, but are easily obtained by known methods. In section 2 the Main Theorem is reformulated using the notation of section 1 (Theorem 12), which represents a basic step in its proof. The proof of Main Theorem is based on Lemmas 28 and 31. The proof of Lemma 28 is in the part *III* [A3] of the paper. This part of the paper also contains the Lemmas 1, for the proof of the Main Theorem in the part *II* of the paper.

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## 1 Preliminary information

We recall the definition of the cobordism groups of framed immersions in Euclidean space, which is a special case of a more general construction presented in the book [K1] on page 55 and in section 10. The connection with the Pontryagin-Thom construction is explained in [A-E].

Let  $f : M^{n-1} \looparrowright \mathbb{R}^n$  be a smooth immersion, where the  $(n-1)$ -dimensional manifold  $M^{n-1}$  is closed, but in general is nonorientable and nonconnected. We introduce a relation of cobordism on the space of such immersions. We say that two immersions  $f_0, f_1$  are connected by a cobordism,  $f_0 \sim f_1$ , if there exists an immersion  $\Phi : (W^n, \partial W = M_0^{n-1} \cup M_1^{n-1}) \looparrowright (\mathbb{R}^n \times [0; 1]; \mathbb{R}^n \times \{0, 1\})$  satisfying the boundary conditions  $f_i = \Phi|_{M_i^{n-1}} : M_i^{n-1} \looparrowright \mathbb{R}^n \times \{i\}$ ,  $i = 0, 1$  and, moreover, it is required that the immersion  $\Phi$  is orthogonal to  $\mathbb{R} \times \{0, 1\}$ .

The set of cobordism classes of immersions forms an Abelian group with respect to the operation of disjoint union of immersions. For example, the trivial element of this group is represented by an empty immersion, the element that is inverse to a given element represented by an immersion  $f_0$  is represented by the composition  $S \circ f_0$ , where  $S$  is a mirror symmetry of the space  $\mathbb{R}^n$ .

This group is denoted by  $Imm(n-1, 1)$ . Because  $\mathbb{R}P^\infty = MO(1)$ , by the Wales theorem [Wa] (see [E1], [Sz2] for references) this group maps onto the stable homotopy group of spheres  $\lim_{k \rightarrow \infty} \pi_{n+k}(S^k)$ .

The immersion  $f$  defines an isomorphism of the normal bundle to the manifold  $M^{n-1}$  and the orientation line bundle  $\kappa$ , i.e. an isomorphism  $D(f) : T(M^{n-1}) \oplus \kappa \cong n\varepsilon$ , where  $\varepsilon$  is the trivial line bundle over  $M^{n-1}$ . In similar constructions in surgery theory of smooth immersions one requires a stable isomorphism of the normal bundle of a manifold  $M^{n-1}$  and the orientation line bundle  $\kappa$ , i.e. an isomorphism  $T(M^{n-1}) \oplus \kappa \oplus N\varepsilon \cong (n+N)\varepsilon$ , for  $N > 0$ . Using Hirsch's Theorem [Hi], it is easy to verify that if two immersions of  $f_1, f_2$  determine isomorphisms  $D(f_1), D(f_2)$ , that belong to the same class of stable isomorphisms then the immersions  $f_1, f_2$  are regularly cobordant and even regularly concordant (but, generally speaking, may not be regularly homotopic).

We also require groups  $Imm^{sf}(n-k, k)$ . An element of this group is represented by a triple  $(f, \kappa, \Xi)$ , where  $f : M^{n-k} \looparrowright \mathbb{R}^n$  is an immersion of a closed manifold,  $\kappa : E(\kappa) \rightarrow M^{n-k}$  is a line bundle (to shorten notation, we shall denote the line (one-dimensional) bundle and its characteristic class in  $H^1(M^{n-k}; \mathbb{Z}/2)$  by the same symbol), and  $\Xi$  is a skew-framing of the normal bundle of the immersion by means of the line bundle  $\kappa$ , i.e., an isomorphism of the normal bundle of the immersion  $f$  and the bundle  $k\kappa$ . In the case of odd  $k$ , the line bundle  $\kappa$  turns out to be orientable over  $M^{n-k}$  and necessarily  $\kappa = w_1(M^{n-k})$ .

Two elements of the cobordism group, represented by triples  $(f_1, \kappa_1, \Xi_1), (f_2, \kappa_2, \Xi_2)$  are equal if the immersions  $f_1, f_2$  are cobordant (this definition is analogous to the previous one for representatives of the group  $Imm^{sf}(n-1, 1)$ ), where in addition it is required that the immersion of the cobordism

be skew-framed, and that the skew-framing of the cobordism be compatible with the given skew-framings on the components of the boundary. We remark that for  $k = 1$  the new definition of the group  $Imm^{sf}(n - k, k)$  coincides with the original definition.

We define a homomorphism

$$J^{sf} : Imm^{sf}(n - 1, 1) \rightarrow Imm^{sf}(n - k, k),$$

which is called the homomorphism of transition to codimension  $k$ . Consider a manifold  $M^{m-1}$  and an immersion  $f' : M^{m-1} \looparrowright \mathbb{R}^n$  representing an element of the first group, and consider a classifying map  $\kappa' : M^{m-1} \rightarrow \mathbb{R}P^a$  to a real projective space of large dimension ( $a = n - 1$  suffices), representing the first Stiefel-Whitney class  $w_1(M^{m-1})$ . Consider the standard subspace  $\mathbb{R}P^{a-k+1} \subset \mathbb{R}P^a$  of codimension  $(k - 1)$ . Assume that the mapping  $\kappa'$  is transverse along the chosen subspace and define the submanifold  $M^{n-k} \subset M^{m-1}$  as the complete inverse image of this subspace for our mapping,  $M^{n-k} = \kappa'^{-1}(\mathbb{R}P^{a-k+1})$ . Define an immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$  as the restriction of the immersion  $f'$  to the given submanifold. Notice that the immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$  admits a natural skew-framing. In fact, the normal bundle to the submanifold  $M^{n-k} \subset M^{m-1}$  is naturally isomorphic to the bundle  $(k - 1)\kappa$ , where  $\kappa = \kappa'|_M$  (here and below, when a manifold is used as a subscript, the superscript indicating the dimension of the manifold is omitted). The isomorphism  $\Xi$  is defined by the standard skew-framing of the normal bundle to the submanifold  $\mathbb{R}P^{a-k+1}$  of the manifold  $\mathbb{R}P^a$ , which is transported to the submanifold  $M^{n-k} \subset M^{m-1}$ , since it is assumed that  $\kappa'$  is transverse regular along  $\mathbb{R}P^{a-k+1}$ . A further direct summand in the skew-framing of the normal bundle of the immersion  $f$  corresponds to the normal line bundle of the immersion  $f'$ . This bundle serves as orientation bundle for  $M^{m-1}$ , hence its restriction to  $M^{n-k}$  coincides with  $\kappa$ . The homomorphism  $J^{sf}$  carries the element represented by the immersion  $f'$  to the element represented by the triple  $(f, \kappa, \Xi)$ . Elementary geometrical considerations, using only the concept of transversality imply that the homomorphism  $J^{sf}$  is correctly defined.

We now define the manifold of double points of self-intersection of an immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$  in general position, and a canonical 2-sheeted covering over this manifold. Under the assumption that the immersion  $f$  is in general position, the subset in  $\mathbb{R}^n$  of points of self-intersection of the immersion  $f$  is denoted by  $\Delta = \Delta(f)$ ,  $\dim(\Delta) = n - 2k$ . This subset is defined by the formula

$$\Delta = \{x \in \mathbb{R}^n : \exists x_1, x_2 \in M^{n-k}, x_1 \neq x_2, f(x_1) = f(x_2) = x\}, \quad (2)$$

We define  $\bar{\Delta} \subset M^{n-k}$  by the formula  $\bar{\Delta} = f^{-1}(\Delta)$ .

We recall the standard definition of the manifold of points of self-intersection and the parameterizing immersion, see e.g. [Ada] for details.

**Definition of self-intersection manifold**

The set  $N$  is defined by the formula

$$N = \{[(x_1, x_2)] \in (M^{n-k} \times M^{n-k})/T' : x_1 \neq x_2, f(x_1) = f(x_2)\} \quad (3)$$

( $T'$  is the involution permuting the coordinate factors), and its canonical covering is defined by the formula

$$\bar{N} = \{(x_1, x_2) \in M^{n-k} \times M^{n-k} : x_1 \neq x_2, f(x_1) = f(x_2)\}. \quad (4)$$

Under the assumption that the immersion  $f$  is generic,  $N$  is a smooth manifold of dimension  $\dim(N) = n - 2k$ . This manifold is denoted by  $N^{n-2k}$  and is called the self-intersection manifold of  $f$ , the projection of the covering is denoted by  $p : \bar{N}^{n-2k} \rightarrow N^{n-2k}$  and is called the canonical 2-sheeted covering.

The immersion  $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$ , parameterizing  $\bar{\Delta}$ , is defined by the formula  $\bar{g} = p|_{\bar{N}}$ . The immersion

$$g : N^{n-2k} \looparrowright \mathbb{R}^n, \quad (5)$$

which is a parametrization of  $\Delta$ , is defined by the formula  $g([x_1, x_2]) = f(x_1)$ . Notice that the parameterizing immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  of  $\Delta$  in general, is not an immersion in general position. There is a two-sheeted covering  $p : \bar{N}^{n-2k} \rightarrow N^{n-2k}$ , for which  $g \circ p = f \circ \bar{g}$ . This covering is called the canonical covering over the manifold of points of self-intersection.

**Definition 1.** Let  $(f, \kappa, \Xi)$  represent an element in the group  $Imm^{sf}(n - k, k)$ . Let us define the homomorphism:

$$h_k : Imm^{sf}(n - k, k) \rightarrow \mathbb{Z}/2,$$

called the stable Hopf invariant by the following formula:

$$h_k([f, \kappa, \Xi]) = \langle \kappa^{n-k}; [M^{n-k}] \rangle.$$

The definitions of the stable Hopf invariant (in the sense of Definition 1) for distinct values of  $k$  are compatible with one another, and coincide with the definition used in the introduction. We formulate this as a separate assertion.

**Proposition 2.** *The homomorphism  $J^{sf} : Imm^{sf}(n-1, 1) \rightarrow Imm^{sf}(n-k, k)$  preserves the Hopf invariant, i.e. the invariant  $h_1 : Imm^{sf}(n-1, 1) \rightarrow \mathbb{Z}/2$  and the invariant  $h_k : Imm^{sf}(n-k, k) \rightarrow \mathbb{Z}/2$  are related by the formula:*

$$h_1 = h_k \circ J^{sf}. \quad (6)$$

### Proof of Proposition 2

Let  $f : M^{n-k} \looparrowright \mathbb{R}^n$  be an immersion with a skew-framing  $\Xi$  of its normal bundle and with characteristic class  $\kappa \in H^1(M^{n-k}; \mathbb{Z}/2)$ , representing an element of  $Imm^{sf}(n-k, k)$ , which satisfies  $J^{sf}([f']) = [f, \kappa, \Xi]$  for an element  $[f'] \in Imm^{sf}(n-1, 1)$ , where  $f$  and  $f'$  are related as in the definition of  $J^{sf}$ . By definition,  $h_k([f, \kappa, \Xi]) = \langle \kappa^{n-k}; [M^{n-k}] \rangle$ .

On the other hand,  $M^{n-k} \subset M'^{n-1}$  is a cycle dual in the sense of Poincaré to the cohomology class  $\kappa'^{k-1} \in H^{k-1}(M'^{n-1}; \mathbb{Z}/2)$ . The formula (6) is valid, since  $\langle \kappa'^{n-1}; [M'^{n-1}] \rangle = \langle \kappa^{n-k}; [M^{n-k}] \rangle$ . In the case  $k = n$  this formula is also satisfied. Proposition 2 is proved.

Let us present an alternative proof. The image of the characteristic mapping  $\kappa' : M'^{n-1} \rightarrow \mathbb{RP}^\infty$ , without loss of generality lies in the skeleton of dimension  $n-1$  ( $n-k$ ) of the classifying space. This allows us to write  $\kappa' : M'^{n-1} \rightarrow \mathbb{RP}^{n-1}$ .

Moreover, let us assume that the mapping  $\kappa'$  is transversal along  $\mathbb{RP}^{n-k} \subset \mathbb{RP}^{n-1}$ , i.e.  $M^{n-k} \subset M'^{n-1}$  is determined by the formula  $M^{n-k} = \kappa'^{-1}(\mathbb{RP}^{n-k} \subset \mathbb{RP}^{n-1})$ ,  $\kappa = \kappa'|_{M^{n-k}}$ . The marked point  $pt \in \mathbb{RP}^{n-k} \subset \mathbb{RP}^{n-1}$  is a regular value for  $\kappa$ .

The characteristic number  $\langle \kappa'^{n-1}; [M'^{n-1}] \rangle$  ( $\langle \kappa^{n-k}; [M^{n-k}] \rangle$ ) coincides with the degree  $\deg(\kappa')$  ( $\deg(\kappa)$ ) of the classifying map  $\kappa' : M'^{n-1} \rightarrow \mathbb{RP}^{n-1}$  ( $\kappa : M^{n-k} \rightarrow \mathbb{RP}^{n-k}$ ), which is considered modulo 2 and is determined as the parity of the number of preimages of regular values of this map. The value  $\deg(\kappa')$  ( $\deg(\kappa)$ ) does not depend on the choice of the mapping  $\kappa'$  ( $\kappa$ ) to the skeleton, as above. The degrees  $\deg(\kappa')$  and  $\deg(\kappa)$  coincide, since the regular value can be chosen common to these mappings. Proposition 2 is alternatively proved.

Let us formulate another (equivalent) definition of the stable Hopf invariant (assuming  $n-2k > 0$ ).

**Definition 3.** Let  $(f, \kappa, \Xi)$  represent an element in the group  $Imm^{sf}(n-k, k)$ ,  $n-2k > 0$ . Let  $N^{n-2k}$  be the manifold of the double points of the



immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$ ,  $\bar{N}$  be the canonical 2-sheeted cover over  $N$ ,  $\kappa_{\bar{N}} \in H^1(\bar{N}; \mathbb{Z}/2)$  be induced from  $\kappa \in H^1(M^{n-k}; \mathbb{Z}/2)$  by the immersion  $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$ .

Let us define the homomorphism  $h_k^{sf} : Imm^{sf}(n-k, k) \rightarrow \mathbb{Z}/2$  by the formula:

$$h_k^{sf}([f, \kappa, \Xi]) = \langle \kappa_{\bar{N}}^{n-2k}, [\bar{N}^{n-2k}] \rangle.$$

The following proposition establishes the equivalence of Definitions 1 and 3.

**Proposition 4.** *Let us assume that the conditions of Definition 3 are satisfied. Then we have:*

$$\langle \kappa_{\bar{N}}^{n-2k}, [\bar{N}^{n-2k}] \rangle = \langle \kappa^{n-k}, [M^{n-k}] \rangle. \quad (7)$$

#### Proof of Proposition 4

Let  $f : M^{n-k} \looparrowright \mathbb{R}^n$  be an immersion with a skew framing  $\Xi$  and with the characteristic class  $\kappa \in H^1(M^{n-k}; \mathbb{Z}/2)$ ; then the triple  $(f, \Xi, \kappa)$  represents an element in the group  $Imm^{sf}(n-k, k)$ . Let  $N^{n-2k}$  be the manifold of self-intersection points of the immersion  $f$ ,  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  be the parameterizing immersion, and  $\bar{N}^{n-2k} \rightarrow N^{n-2k}$  be the canonical double covering. Consider the image of the fundamental class  $\bar{g}_*([\bar{N}^{n-2k}]) \in H_{n-2k}(M^{n-k}; \mathbb{Z}/2)$  by the immersion  $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$  and denote by  $m \in H^k(M^{n-k}; \mathbb{Z}/2)$  the cohomology class Poincaré dual to the homology class  $\bar{g}_*([\bar{N}^{n-2k}])$ . Consider also the cohomology Euler class of the normal bundle immersion  $f$ , which is denoted by  $e \in H^k(M^{n-k}; \mathbb{Z}/2)$ .

By the Herbert Theorem for the immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$  with self-intersection manifold  $N^{n-2k}$  (see [E-G], Theorem 1.1 the case  $r = 1$  coefficients is  $\mathbb{Z}/2$ ; see also this theorem in the original papers [He], [L-S]) the following formula is valid:

$$e + m = 0. \quad (8)$$

Since the Euler class  $e$  of the normal bundle  $k\kappa$  of the immersion  $f$  is equal to  $\kappa^k$  (line bundles and their corresponding characteristic cohomology classes are denoted by the same symbols), then the cycle  $\bar{g}_*([\bar{N}]) \in H_{n-2k}(M^{n-k}; \mathbb{Z}/2)$  is Poincaré dual to the cohomology class  $\kappa^k \in H^k(M^{n-k}; \mathbb{Z}/2)$ . Therefore, the formula (7) and Proposition 4 are proved.

It is more convenient to reformulate Proposition 4 (in a more general form) by means of the language of commutative diagrams. The desired reformulation is given in Lemma 7 below. We turn to the relevant definitions.

Let  $g : N^{n-2} \looparrowright \mathbb{R}^n$  be the immersion of the double self-intersection points of the immersion  $f : M^{n-1} \looparrowright \mathbb{R}^n$  of codimension 1. We denote by  $\nu_N : E(\nu_N) \rightarrow N^{n-2}$  the normal 2-dimensional bundle of the immersion  $g$ . (Note that the disk bundle associated with the vector bundle  $\nu_N$  is diffeomorphic to a regular closed tubular neighborhood of the immersion  $g$ .)

In comparison with an arbitrary vector bundle, this bundle carries an additional structure, namely its structure group as an  $O(2)$ -bundle admits a reduction to a discrete dihedral group which we denote by  $\mathbf{D}$ . This group has order 8, and is defined as the group of orthogonal transformations of the plane which carry the standard pair of coordinate axes into themselves (with possible change of orientation and order).

In the standard presentation of the group  $\mathbf{D}$  there are two generators  $a, b$  which are connected by the relations  $\{a^4 = b^2 = 1, [a, b] = a^2\}$ . The generator  $a$  is represented by the rotation of the plane through an angle  $\frac{\pi}{2}$ , and the generator  $b$  is represented by a reflection with respect to the bisector of the first and second coordinate axes. Notice that the element  $ba$  (the product means the rule of composition  $b \circ a$  of transformations in  $O(2)$ ) is represented by the reflection with respect to the first coordinate axis.

### **The structure group of the normal bundle of the manifold of self intersection points for an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ , in case $k = 1$**

Let us use the transversality condition for the immersion  $f : M^{n-1} \looparrowright \mathbb{R}^n$ . Let  $N^{n-2}$  be the self-intersection manifold of the immersion  $f, g : N^{n-2} \looparrowright \mathbb{R}^n$  be the parameterizing immersion. In the fiber  $E(\nu_N)_x$  of the normal bundle  $\nu_N$  over the point  $x \in N$  an unordered pair of axes is fixed. These axes are formed by the tangents to the curves of intersection of the fiber  $E(\nu_N)_x$  with two sheets of immersed manifolds intersecting transversely in the neighborhoods of this point. By construction, the bundle  $\nu_N$  has the structure group  $\mathbf{D} \subset O(2)$ .

Over the space  $K(\mathbf{D}, 1)$  the universal 2-dimensional  $\mathbf{D}$ -bundle is defined. This bundle will be denoted by  $\psi : E(\psi) \rightarrow K(\mathbf{D}, 1)$ . We say that the mapping  $\eta : N \rightarrow K(\mathbf{D}, 1)$  is classifying for the bundle  $\nu_N$ , if an isomorphism  $\Xi : \eta^*(\psi) \cong \nu_N$  is well defined, where  $\eta^*(\psi)$  is the inverse image of the bundle  $\psi$  and  $\nu_N$  is the normal bundle of the immersion  $g$ . Further a bundle itself and its classifying map will be denoted the same; in the considered case

we have  $\eta \cong \nu_N$ . The isomorphism  $\Xi$  will be called a **D**-framing of the immersion  $g$ , and the mapping  $\eta$  will be called the characteristic mapping of the **D**-framing  $\Xi$ .

**Remark.** In fact, we have described only part of a more general construction. The structure group of the  $s$ -dimensional normal bundle to the submanifold  $N_s$  of points of self-intersection of multiplicity  $s$  of an immersion  $f$  admits a reduction to the structure group  $\mathbb{Z}/2 \wr \Sigma(s)$ , the wreath product of the cyclic group  $\mathbb{Z}/2$  with the group of permutations of a set of  $s$  elements (cf., for example, [E1]).

Let a triple  $(f, \kappa, \Xi)$ , where  $f : M^{n-k} \looparrowright \mathbb{R}^n$  is an immersion and  $\Xi$  is a skew-framing of  $f$  with the characteristic class  $\kappa \in H^1(M; \mathbb{Z}/2)$ , represent an element of the group  $Imm^{sf}(n-k, k)$ . We need to generalize the previous construction for  $k = 1$  and to describe the structure group of the normal bundle  $\nu_N$  to the manifold  $N^{n-2k}$  of self-intersection points of a generic immersion of an arbitrary codimension  $k$ ,  $k \leq [\frac{n}{2}]$ .

**Proposition 5.** *The normal bundle  $\nu_N$ ,  $\dim(\nu_N) = 2k$ , of the immersion  $g$  is a direct sum of  $k$  copies of a two-plane bundle  $\eta$  over  $N^{n-2k}$ , where each two-plane bundle has structure group **D** and is classified by a classifying mapping  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  ( an analogous proposition is proved in [Sz2]).*

### Proof of Proposition 5

Let  $x \in N^{n-2k}$  be a point in the manifold of double points. Denote by  $\bar{x}_1, \bar{x}_2 \in \bar{N}^{n-2k} \looparrowright M^{n-k}$  the two preimages of this point under the canonical covering map by the double covering. The orthogonal complement in the space  $T_{g(x)}(\mathbb{R}^n)$  to the subspace  $g_*(T_x(N^{n-2k}))$  is the fiber of the normal bundle  $E(\nu_N)$  of the immersion  $g$  over a point  $x \in N^{n-2k}$ . This fiber is represented as a direct sum of two linear spaces,  $E(\nu_N)_x = \bar{E}_{x,1} \oplus \bar{E}_{x,2}$ , where each subspace  $\bar{E}_{x,i} \subset E(\nu_N)_x$  is a fiber of the normal bundle of the immersion  $f$  at the point  $\bar{x}_i$ .

Each subspace  $\bar{E}_{x,i}$  of the fiber is canonically a direct sum of  $k$  ordered copies of the fiber of a line bundle, since the normal bundle to the immersion  $f$  is equipped with a skew-framing. We group the fibers  $\bar{E}(\kappa_{x,j,i})$ ,  $j = 1, \dots, k$ ,  $i = 1, 2$  with a corresponding index into a two-dimensional subfiber of the fiber of the normal bundle  $E(\nu_N)_x$ . As a result, we obtain a decomposition of the fiber  $E(\nu_N)_x$  over each point  $x \in N^{n-2k}$  into a direct sum of  $k$  copies of a two-dimensional subspace. This construction depends continuously on the

choice of the point  $x$ , and can be carried out simultaneously for each point of the base  $N^{n-2k}$ . As a result, we obtain the required decomposition of the bundle  $\nu_N$  into a direct sum of a number of canonically isomorphic two-plane bundles. Each two-dimensional summand is classified by a structure map  $\eta : N \rightarrow K(\mathbf{D}, 1)$ , which proves Proposition 5.

**Definition 6.** We define the cobordism group of immersions  $Imm^{\mathbf{D}}(n - 2k, 2k)$ , assuming  $n > 2k$ . Let  $(g, \eta, \Psi)$  be a triple, which determines a  $\mathbf{D}$ -framed immersion of codimension  $2k$ . Here  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  is an immersion and  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  is the classifying map of the  $\mathbf{D}$ -framing  $\Psi$ . The cobordism relation of triples is standard.

**Lemma 7.** *Under the assumption  $k_1 < k$ ,  $2k < n$ , the following commutative diagram of groups is well defined:*

$$\begin{array}{ccccc} Imm^{sf}(n - k_1, k_1) & \xrightarrow{J^{sf}} & Imm^{sf}(n - k, k) & \xrightarrow{h_k^{sf}} & \mathbb{Z}/2 \\ \downarrow \delta_{k_1} & & \downarrow \delta_k & & \parallel \\ Imm^{\mathbf{D}}(n - 2k_1, 2k_1) & \xrightarrow{J^{\mathbf{D}}} & Imm^{\mathbf{D}}(n - 2k, 2k) & \xrightarrow{h_k^{\mathbf{D}}} & \mathbb{Z}/2. \end{array} \quad (9)$$

### Proof of Lemma 7

We define the homomorphisms in the diagram (9). The homomorphism

$$Imm^{sf}(n - k_1, k_1) \xrightarrow{J^{sf}} Imm^{sf}(n - k, k)$$

is defined exactly as the homomorphism  $J_{sf}$  for the case  $k_1 = 1$ .

Define the homomorphism

$$Imm^{\mathbf{D}}(n - 2k_1, 2k_1) \xrightarrow{J^{\mathbf{D}}} Imm^{\mathbf{D}}(n - 2k, 2k) \quad (10)$$

Let us present 3 (equivalent) definitions. Assume that a triple  $[(g', \eta', \Psi')]$  represents an element in the cobordism group  $Imm^{\mathbf{D}}(n - 2k, 2k)$ , where  $g' : N^{n-2k} \looparrowright \mathbb{R}^n$  is an immersion, which is equipped with the dihedral framing.

–1. Take the universal  $\mathbf{D}$ -bundle  $\psi$  over  $K(\mathbf{D}, 1)$  and take the pull-back  $\eta'^*(\psi)$  of this bundle by means of the classifying map  $\eta'$ . Take a submanifold  $N^{n-2k_1} \subset N^{n-2k}$ , which represents the Euler class of the bundle  $(k - k_1)\eta'^*(\psi)$ . The triple  $(g, \eta, \Psi)$  is well defined, where  $g = g'|_N$  and  $\eta = \eta'|_N$ . The  $\mathbf{D}$ -framing  $\Psi$  is defined below.

Let us consider the normal bundle  $\nu_{g'}$  of the immersion  $g'$ . The restriction of this bundle on the submanifold  $N^{n-2k} \subset N^{m-2k_1}$  is decomposed into the Whitney sum of the two bundles:  $\nu_{g'} = \nu_g|_N \oplus \nu_{N \subset N'}$ , where  $\nu_{N \subset N'}$  is the normal bundle of the submanifold  $N^{n-2k} \subset N^{m-2k_1}$ . The bundles  $\nu_{N \subset N'}$  and  $(k - k_1)\eta^*(\psi)$  are isomorphic and this bundle is equipped with the standard  $\mathbf{D}$ -framing. Therefore the bundle  $\nu_g$  is equipped with the dihedral framing  $\Psi : \nu_g \cong k\eta^*(\psi)$ . The triple  $(g, \eta, \Psi)$  represent the required element  $J^{\mathbf{D}}(g', \eta', \Psi') \in Imm^{\mathbf{D}}(n - 2k, 2k)$ .

–2. Consider the configuration space  $Sym_2(\mathbb{RP}^s)$  of non-ordered pairs of distinguished points in  $\mathbb{RP}^s$ , let us denote this configuration space by  $\Gamma_o(s)$  (see the formula (42), in which we assume that  $s = n - k$ ) in the case  $s > 2(n - k_1) + 1$ . The space  $\Gamma_o(s)$  is an open  $2s$ -dimensional manifold, with the homotopy  $n - k$ -type of the space  $K(\mathbf{D}, 1)$ . Without loss of a generality we may assume that the characteristic mapping is the following:  $\eta' : N^{m-2k_1} \rightarrow \Gamma_o(s)$ . Let us consider the submanifold  $\Gamma_o(s - k + k_1) \subset \Gamma_o(s)$  of the codimension  $2(k - k_1)$ , which is induced by the standard inclusion  $\mathbb{RP}^{s-k+k_1} \subset \mathbb{RP}^s$ . Assume, without loss of the generality, that the mapping  $\eta'$  is transversal along the submanifold  $\Gamma_o(s - k + k_1) \subset \Gamma_o(s)$ . Define a submanifold  $N^{n-2k} \subset N^{m-2k_1}$  by the formula  $N^{n-2k} = \eta'^{-1}(\Gamma_o(s - k + k_1))$ . Obviously, the normal bundle of the submanifold  $\Gamma_o(s - k + k_1) \subset \Gamma_o(s)$  is isomorphic to the bundle  $(k - k_1)\psi$ . Therefore the bundles  $\nu_{N \subset N'}$  and  $(k - k_1)\eta^*(\psi)$  are isomorphic. Let us define a triple  $(g, \eta, \Psi)$ , which represents the element  $J^{\mathbf{D}}(g', \eta', \Psi') \in Imm^{\mathbf{D}}(n - 2k, 2k)$ , analogously to the Definition –1.

–3. Let us consider the canonical 2-sheeted covering  $\bar{N}^{m-2k_1} \rightarrow N^{m-2k_1}$ , which is induced from the universal 2-sheeted covering  $K(\mathbf{I}_c, 1) \rightarrow K(\mathbf{D}, 1)$  by the mapping  $\eta'$  (the subgroup  $\mathbf{I}_c \subset \mathbf{D}$  of the index 2 is defined below by the formula (14)). Let us denote the involution of this 2-sheeted covering by  $T : \bar{N}^{m-2k_1} \rightarrow \bar{N}^{m-2k_1}$ . Consider the submanifold  $\bar{W}^{n-k-k_1} \subset \bar{N}^{m-2k_1}$  of the codimension  $k - k_1$ , which is defined as the transversal preimage of the submanifold  $\mathbb{RP}^{s-k+k_1} \subset \mathbb{RP}^s$  by the mapping  $\bar{N}^{m-2k_1} \rightarrow \mathbb{RP}^s$  in the standard skeleton  $\mathbb{RP}^s \subset K(\mathbb{Z}/2, 1)$ . Define the mapping  $\bar{N}^{m-2k_1} \rightarrow \mathbb{RP}^s$  by the composition of the canonical 2-sheeted covering mapping  $\bar{\eta}' : \bar{N}^{m-2k_1} \rightarrow K(\mathbf{I}_c, 1)$  over the mapping  $\eta'$  with the mapping  $K(\mathbf{I}_c, 1) \rightarrow K(\mathbb{Z}/2, 1)$ , the last mapping is induced by the epimorphism (15), which is defined below. Let us consider the manifold  $\bar{W}^{n-k-k_1} \cap T(\bar{W}^{n-k-k_1}) \subset \bar{N}^{m-2k_1}$ , assuming that the manifolds  $\bar{W}^{n-k-k_1}$  and  $T(\bar{W}^{n-k-k_1})$  are transversally intersected inside  $\bar{N}^{m-2k_1}$ . Define the manifold  $N^{n-2k}$  as the quotient  $(\bar{W}^{n-k-k_1} \cap T(\bar{W}^{n-k-k_1}))/T$ , this manifold is equipped by the natural embedding into the manifold  $N^{m-2k_1}$ . The normal bundle of the submanifold  $N^{n-2k} \subset N^{m-2k_1}$  is naturally isomorphic to the Whitney sum of  $k - k_1$  copies of a 2-dimensional  $\mathbf{D}$ -bundle with

the classifying mapping  $\eta'|_{N \subset N'}$ . The triple  $(g, \eta, \Psi)$ , which represents the element  $J^{\mathbf{D}}(g', \eta', \Psi') \in Imm^{\mathbf{D}}(n-2k, 2k)$  is defined as in the Definition -1.

Let us prove that Definition -1 and Definition -3 are equivalent. Let us consider the canonical double covering  $p' : \bar{N}^{n-2k_1} \rightarrow N^{n-2k_1}$  over the self-intersection points manifold of  $g'$ . The manifold  $\bar{N}^{n-2k_1}$  is naturally immersed into  $M^{n-k}$ :  $\bar{N}^{n-2k} \looparrowright M^{n-k}$ . Let us consider the submanifold  $M^{n-k} \subset M^{n-k_1}$ , dual to  $\kappa'^{k-k_1}$ . This manifold is used in the definition of the homomorphism  $J^{sf}$ . Let us consider the submanifold  $M^{n-k} \cap \bar{N}^{n-2k_1} \subset \bar{N}^{n-2k_1}$ , assuming that  $M^{n-k}$  intersects the immersion  $\bar{N}^{n-2k_1} \looparrowright M^{n-k}$  in a general position. We denote this submanifold by  $\tilde{W}$ . Obviously,  $\dim(\tilde{W}) = n - k + k_1$  and the codimension of the submanifold  $\tilde{W}^{n-k-k_1} \subset \bar{N}^{n-2k_1}$  is equal to  $k - k_1$ .

Let us denote by  $T : \bar{N}^{n-2k} \rightarrow \bar{N}^{n-2k}$  the involution on the covering space of  $p'$ . Let us consider the manifold  $T(\tilde{W}^{n-k+k_1})$  and the intersection  $T(\tilde{W}^{n-k+k_1}) \cap \tilde{W}^{n-k+k_1}$  inside  $\bar{N}^{n-2k_1}$ . Assuming that this intersection is generic,  $T(\tilde{W}^{n-k+k_1}) \cap \tilde{W}^{n-k+k_1}$  is a smooth closed manifold, let us denote this manifold by  $\bar{N}$ ,  $\dim(\bar{N}) = n - 2k$ . Moreover, the manifold  $\bar{N}^{n-2k}$  is equivariant with respect to the involution  $T = T'|_{\bar{N}^{n-2k}}$ . The factor-space  $\bar{N}^{n-2k}/T$  is well defined, this is smooth closed manifold. Let us denote this manifold by  $N^{n-2k}$  and the restriction of the canonical double cover over this manifold by  $p : \bar{N}^{n-2k} \rightarrow N^{n-2k}$ .

Note that the manifold  $N^{n-2k}$  is a submanifold in  $N^{n-2k_1}$  and this submanifold coincides with the self-intersection manifold of the immersion  $f = f'|_{M^{n-k} \subset M^{n-k_1}}$ . The manifold  $N^{n-2k}$  was used in Definition -3 of the homomorphism  $J^{\mathbf{D}}$ .

Let us denote by  $N_2^{n-2k} \subset N^{n-2k_1}$  the manifold, which represents the Euler class of the bundle  $\eta'^*(\psi)$ . This submanifold is well-defined up to regular  $\mathbf{D}$ -framed cobordism. Let us prove that the submanifold  $N^{n-2k} \subset N^{n-2k_1}$  also represents the Euler class of the bundle  $\eta'^*(\psi)$ . Therefore we may put  $N_2^{n-2k} = N^{n-2k}$ .

Обозначим расслоение  $\eta'^*(\psi)$  через  $\xi'$  для краткости. Рассмотрим расслоение  $p'^*(\xi')$ , обозначим это расслоение через  $\bar{\xi}'$ . Рассмотрим расслоение  $T^*(p'^*(\xi'))$ , которое определено как индуцированное из  $\bar{\xi}'$  при инволюции. Очевидно, что расслоение  $\bar{\xi}'$  представлено в виде суммы Уитни двух  $k - k_1$ -мерных расслоений:

$$\bar{\xi}' = \bar{\xi}'_+ \oplus \bar{\xi}'_- . \quad (11)$$

В этой формуле расслоение  $\bar{\xi}'_+$  определяется следующим способом. Рассмотрим универсальное  $\mathbf{D}$ -расслоение  $\psi$  и рассмотрим  $\mathbf{I}_c$ -расслоение  $\bar{\psi}$ , которое индуцировано двулиственным накрытием по подгруппе  $\mathbf{I}_c \subset \mathbf{D}$

индекса 2 (см. формулу (14) ниже). Поскольку  $\mathbf{I}_c \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , расслоение  $\bar{\psi}$  естественно изоморфно сумме Уитни  $\bar{\psi} = \bar{\psi}_+ \oplus \bar{\psi}_-$  двух линейных  $\mathbb{Z}/2$ -расслоений. Сумма (12) индуцирована из указанной.

Let us denote the bundle  $\eta'^*(\psi)$  by  $\xi'$  for short. Take the bundle  $p'^*(\xi')$ , let us denote this bundle by  $\bar{\xi}'$ , and take the bundle  $T^*(p'^*(\xi'))$ , which is induced from  $\bar{\xi}'$  by the involution. Obviously, the bundle  $\bar{\xi}'$  decomposes into the Whitney sum of the two  $k - k_1$ -dimensional bundles:

$$\bar{\xi}' = \bar{\xi}'_+ \oplus \bar{\xi}'_- . \quad (12)$$

In this formula the bundle  $\bar{\xi}'_+$  is defined as follows. Take the universal  $\mathbf{D}$ -bundle  $\psi$  and consider  $\mathbf{I}_c$ -bundle  $\bar{\psi}$ , which is induced by 2-sheeted covering constructed by the index 2 subgroup  $\mathbf{I}_c \subset \mathbf{D}$  (see the formula (14) below). Because  $\mathbf{I}_c \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , the bundle  $\bar{\psi}$  is naturally isomorphic to the Whitney sum  $\bar{\psi} = \bar{\psi}_+ \oplus \bar{\psi}_-$  of the two line  $\mathbb{Z}/2$ -bundles. This Whitney sum induces the Whitney sum (12).

Analogously, we get  $T^*(\bar{\xi}') = T^*(\bar{\xi}'_+) \oplus T^*(\bar{\xi}'_-)$ . Moreover, the following equality is satisfied:  $T^*(\bar{\xi}'_+) = \bar{\xi}'_-$ ,  $T^*(\bar{\xi}'_-) = \bar{\xi}'_+$ . The bundle  $\bar{\xi}'_+$  is isomorphic to the Whitney sum of  $k - k_1$  copies of the line bundle  $\kappa'$ .

The submanifold  $\bar{N}_2^{n-2k} \subset \bar{N}^{m-2k_1}$  is well defined as the restriction of the covering  $p'$  to the submanifold  $N_2^{n-2k} \subset N^{m-2k_1}$ . This submanifold represents the equivariant Euler class of the bundle  $\bar{\xi}'$ . This submanifold  $\bar{N}_2^{n-2k}$  is well defined as the intersection of the two  $n - k + k_1$ -dimensional submanifolds, which will be defined below and which are denoted by  $\bar{W}_{2,+}^{n-k+k_1}$  and  $\bar{W}_{2,-}^{n-k+k_1}$ . The submanifold  $\bar{W}_{2,+}^{n-k+k_1} \subset \bar{N}^{m-2k_1}$  represents the Euler class of the bundle  $\bar{\xi}'_+$ . The submanifold  $\bar{W}_{2,-}^{n-k+k_1} \subset \bar{N}^{m-2k_1}$  represents the Euler class of the bundle  $\bar{\xi}'_-$ . Note that the submanifold  $\bar{W}_{2,-}^{n-k+k_1} \subset \bar{N}^{m-2k_1}$  coincides by definition with  $\tilde{W}^{n-k+k_1}$ . The submanifold  $\bar{W}_{2,+}^{n-k+k_1} \subset \bar{N}^{m-2k_1}$  coincides by definition with  $T(\tilde{W}^{n-k+k_1})$ . Therefore  $\bar{N}_2^{n-2k}$  coincides with  $\bar{N}^{n-2k}$  and  $N_2^{n-2k}$  coincides with  $N^{n-2k}$ . This proves the equivalence of Definition -1 and Definition -3 of the homomorphism  $J^{\mathbf{D}}$ .

Recall, that the homomorphism

$$Imm^{sf}(n - k, k) \xrightarrow{\delta_k} Imm^{\mathbf{D}}(n - 2k, 2k)$$

transforms the cobordism class of a triple  $(f, \kappa, \Xi)$  to the cobordism class of the triple  $(g, \eta, \Psi)$ , where  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  is the immersion parameterizing the self-intersection points manifold of the immersion  $f$  (it is assumed that the immersion  $f$  intersects itself transversally),  $\Psi$  is the  $\mathbf{D}$ -framing of the normal bundle of the immersion  $g$ , and  $\eta$  is the classifying map of the  $\mathbf{D}$ -framing  $\Psi$ .

We turn to the definition of the homomorphism

$$Imm^{\mathbf{D}}(n - 2k, 2k) \xrightarrow{h_k^{\mathbf{D}}} \mathbb{Z}/2, \quad (13)$$

which will be called the dihedral Hopf invariant. Define the subgroup

$$\mathbf{I}_c \subset \mathbf{D}, \quad (14)$$

generated by the transformations of the plane that preserve the subspace spanned by each basis vector. The subgroup  $\mathbf{I}_c$  is an elementary abelian 2-group of rank 2. Define the homomorphism

$$l : \mathbf{I}_c \rightarrow \mathbb{Z}/2, \quad (15)$$

by sending an element  $x$  of  $\mathbf{I}_c$  to 0 if  $x$  fixes the first basis vector, and to 1 if  $x$  sends the first basis vector to its negative.

The subgroup (14) has index 2 and the following 2-sheeted covering:

$$K(\mathbf{I}_c, 1) \rightarrow K(\mathbf{D}, 1), \quad (16)$$

induced by this subgroup is well defined.

Denote by

$$\bar{N}^{n-2k} \rightarrow N^{n-2k} \quad (17)$$

the 2-sheeted covering induced by the classifying map  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  from the covering (16). The following characteristic class is well defined:

$$\bar{\eta}_{sf} = l \circ \bar{\eta}_{\mathbf{I}_c} : \bar{N}^{n-2k} \rightarrow K(\mathbb{Z}/2, 1),$$

where  $\bar{\eta}_{\mathbf{I}_c} : \bar{N}^{n-2k} \rightarrow K(\mathbf{I}_c, 1)$  is the double covering over the classifying map  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  induced by the coverings (16), (17) over the target and the source of the map  $\eta$  respectively.

Let us define the homomorphism  $Imm^{\mathbf{D}}(n - 2k, 2k) \xrightarrow{h_k^{\mathbf{D}}} \mathbb{Z}/2$  by the formula:

$$h_k^{\mathbf{D}}([g, \eta, \Psi]) = \langle (\bar{\eta}_{sf})^{n-2k}, [\bar{N}^{n-2k}] \rangle. \quad (18)$$

The diagram (9) is now well defined. Commutativity of the right square of the diagram follows for  $k_1 = 1$  by Proposition 4, and for an arbitrary  $k_1$  the proof is similar. Let us prove the commutativity of the left square of the diagram. Lemma 7 is proved.



We need an equivalent definition of the dihedral Hopf invariant in the case of  $\mathbf{D}$ -framed immersions in the codimension  $2k$ ,  $n - 4k > 0$ . Consider the subgroup of the orthogonal group  $O(4)$  that transforms the set of vectors  $(\pm \mathbf{e}_1, \pm \mathbf{e}_2, \pm \mathbf{e}_3, \pm \mathbf{e}_4)$  of the standard basis into itself, perhaps by changing the direction of some vectors and, moreover, preserving the non-ordered pair of 2-dimensional subspaces  $\text{Lin}(\mathbf{e}_1, \mathbf{e}_2)$ ,  $\text{Lin}(\mathbf{e}_3, \mathbf{e}_4)$  generated by basis vectors  $(\mathbf{e}_1, \mathbf{e}_2)$ ,  $(\mathbf{e}_3, \mathbf{e}_4)$ . Thus these 2-dimensional subspaces may be preserved or interchanged. Denote the subgroup of these transformations by  $\mathbb{Z}/2^{[3]}$ . This group has order  $2^7$ . Define the chain of subgroups of index 2:

$$\mathbf{I}_c \times \mathbf{D} \subset \mathbf{D} \times \mathbf{D} \subset \mathbb{Z}/2^{[3]}. \quad (19)$$

The subgroup  $\mathbf{D} \times \mathbf{D} \subset \mathbb{Z}/2^{[3]}$  is defined as the subgroup of transformations leaving invariant each 2-dimensional subspace  $\text{Lin}(\mathbf{e}_1, \mathbf{e}_2)$ ,  $\text{Lin}(\mathbf{e}_3, \mathbf{e}_4)$  spanned by pairs of vectors  $(\mathbf{e}_1, \mathbf{e}_2)$ ,  $(\mathbf{e}_3, \mathbf{e}_4)$ . This subgroup is isomorphic to a direct product of two copies of  $\mathbf{D}$ , each factor leaving invariant the corresponding 2-dimensional subspace. The subgroup  $\mathbf{I}_c \times \mathbf{D} \subset \mathbf{D} \times \mathbf{D}$  is defined as the subgroup of transformations that leave invariant each linear subspace  $\text{Lin}(\mathbf{e}_1)$ ,  $\text{Lin}(\mathbf{e}_2)$  generated by vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ .

Let the triple  $(g, \eta, \Psi)$  represent an element of  $\text{Imm}^{\mathbf{D}}(n - 2k, 2k)$ , assuming that  $n - 4k > 0$  and that  $g$  is an immersion in general position. Let  $L^{n-4k}$  be the manifold of double self-intersection points of the immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$ . The following tower of 2-sheeted coverings

$$\bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k} \rightarrow \bar{L}_{\mathbf{D} \times \mathbf{D}}^{n-4k} \rightarrow L^{n-4k} \quad (20)$$

is well defined by the following construction. (The covering  $\bar{L}_{\mathbf{D} \times \mathbf{D}}^{n-4k} \rightarrow L^{n-4k}$  was considered above as the canonical covering over self-intersection manifold of the immersion  $g$  and was denoted by  $\bar{L}^{n-4k} \rightarrow L^{n-4k}$ .) Let us consider the parameterizing immersion  $h : L^{n-4k} \looparrowright \mathbb{R}^n$ . The normal bundle of the immersion  $h$  will be denoted by  $\nu_L$ . This bundle is classified by a mapping  $\zeta : L^{n-4k} \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$ .

The chain of subgroups (19) induces a tower of 2-sheeted coverings of classifying spaces:

$$K(\mathbf{I}_c \times \mathbf{D}, 1) \subset K(\mathbf{D} \times \mathbf{D}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1) \quad (21)$$

over the target of the classifying map  $\zeta$  and the tower of 2-sheeted coverings (20) over the domain of the mapping  $\zeta$ . The covering  $\bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k} \rightarrow L^{n-4k}$ , defined by the formula (20), will be called the canonical 4-sheeted covering over the manifold of points of self-intersections of the immersion  $g$ .

Define the epimorphism

$$l^{[3]} : \mathbf{I}_c \times \mathbf{D} \rightarrow \mathbf{I}_d, \quad (22)$$

by sending an element  $x$  of  $\mathbf{I}_c \times \mathbf{D}$  to 0 if  $x$  fixes the first basis vector, and to 1 if  $x$  sends the first basis vector to its negative. This map induces the map of the classifying spaces:

$$K(\mathbf{I}_c \times \mathbf{D}, 1) \rightarrow K(\mathbf{I}_d, 1). \quad (23)$$

The classifying mapping  $\bar{\zeta}_{\mathbf{I}_c \times \mathbf{D}} : \bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k} \rightarrow K(\mathbf{I}_c \times \mathbf{D}, 1)$  is well defined as a result of the transition to a 4-sheeted covering over the mapping  $\zeta$  and the classifying map  $\bar{\zeta}_{sf} : \bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k} \rightarrow K(\mathbf{I}_d, 1)$  is well defined as a result of the composition of the classifying map  $\bar{\zeta}_{\mathbf{I}_c \times \mathbf{D}}$  with the map (23).

**Proposition 8.** *Suppose that a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$  represents an element of  $Imm^{\mathbf{D}}(n-2k, 2k)$ , the following formula is satisfied:*

$$\langle (\bar{\zeta}_{sf})^{n-4k}; [\bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k}] \rangle = \langle (\bar{\eta}_{sf})^{n-2k}; [\bar{N}^{n-2k}] \rangle. \quad (24)$$

### Proof of Proposition 8

Let  $(g : N^{n-2k} \looparrowright \mathbb{R}^n, \Psi, \eta)$  be a  $\mathbf{D}$ -framed immersion, representing an element of  $Imm^{\mathbf{D}}(n-2k, 2k)$  in the image of the homomorphism  $\delta^k$ . Let  $L^{n-4k}$  be the double-points manifold of the immersion  $g$ ,  $h : L^{n-4k} \looparrowright \mathbb{R}^n$  be the parameterizing immersion, and  $\bar{L}^{n-4k} \rightarrow L^{n-4k}$  be the canonical double covering. Consider the image of the fundamental class  $\bar{h}_*([\bar{L}^{n-4k}]) \in H_{n-4k}(N^{n-2k}; \mathbb{Z}/2)$  by means of the immersion  $\bar{h} : \bar{L}^{n-4k} \looparrowright N^{n-2k}$  and let us denote by  $m \in H^{2k}(N^{n-2k}; \mathbb{Z}/2)$  the cohomology class that is Poincaré-dual to the homology class  $\bar{h}_*([\bar{L}^{n-4k}])$ . Consider also the cohomology Euler class of the normal bundle immersion  $g$ , which is denoted by  $e \in H^{2k}(N^{n-2k}; \mathbb{Z}/2)$ .

By the Herbert Theorem (see [E-G], Theorem 1.1, coefficients  $\mathbb{Z}/2$ ) for immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  with the self-intersection manifold  $L^{n-4k}$  the formula  $e = m$  given in (8) is valid. Let us consider the classifying map  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$ . Let us consider the 2-sheeted cover  $K(\mathbf{I}_c, 1) \rightarrow K(\mathbf{D}, 1)$  over the classifying space. Let us induce  $\eta$  the 2-sheeted covering map over the map  $\eta$ , denoted by  $\bar{\eta}_{sf} : \bar{N}_{sf}^{n-2k} \rightarrow K(\mathbf{I}_c, 1)$ . (Note that in the case  $N^{n-2k}$  is a self-intersection points manifold of a skew-framed immersion, the manifold  $\bar{N}_{sf}^{n-2k}$  was considered above and this manifold was called the canonical covering manifold over  $N^{n-2k}$ , this manifold was denoted by  $\bar{N}^{n-2k}$ .)

Let us denote by  $\bar{e} \in H^{2k}(\bar{N}_{sf}^{n-2k}; \mathbb{Z}/2)$ ,  $\bar{m} \in H^{2k}(\bar{N}_{sf}^{n-2k}; \mathbb{Z}/2)$  the images of the cohomology classes  $e, m$ , respectively, under the canonical double cover  $\bar{N}_{sf}^{n-2k} \rightarrow N^{n-2k}$ . The Herbert Theorem implies that:

$$\bar{e} = \bar{m},$$

in particular, the following formula holds:

$$\langle (\bar{\eta}_{sf})^{n-4k} \bar{m}; [\bar{N}_{sf}^{n-2k}] \rangle = \langle (\bar{\eta}^{sf})^{n-4k} \bar{e}; [\bar{N}_{sf}^{n-2k}] \rangle. \quad (25)$$

Because  $\bar{\eta}_{sf}$  coincides with  $\bar{e}$ , the right side of the formula is equal to  $\langle (\bar{\eta}_{sf})^{n-2k}; [\bar{N}^{n-2k}] \rangle$ . Because  $\bar{m}$  is dual to the cohomology class  $\bar{h}_*[\bar{L}_{\mathbf{H}_{\bar{e}}}^{n-4k}]$ , the left side of the formula (25) can be rewritten in the form:  $\langle \bar{\eta}_{sf}^{n-4k}; [\bar{L}_{\mathbf{H}_{\bar{e}}}^{n-2k}] \rangle$ . Because the classifying mappings  $\bar{\zeta}_{sf} : \bar{L}_{\mathbf{H}_{\bar{e}}}^{n-4k} \rightarrow K(\mathbb{Z}/2, 1)$  and  $\bar{\eta}_{sf}|_{\bar{L}_{\mathbf{H}_{\bar{e}}}}$  coincide, the left side of the formula (25) is equal to the characteristic number  $\langle (\bar{\zeta}_{sf})^{n-4k}; [\bar{L}_{\mathbf{H}_{\bar{e}}}^{n-4k}] \rangle$ . Proposition 8 is proved.

Let us generalize Proposition 5, Definition 6, and Lemma 7.

Let us assume that the triple  $(g, \eta, \Psi)$ , where  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  is an immersion,  $\Psi$  is a  $\mathbf{D}$ -framing of the normal bundle of the immersion  $g$  with the characteristic class  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$ , represents an element in  $Imm^{\mathbf{D}}(n-2k, 2k)$ . Let  $h : L^{n-4k} \looparrowright \mathbb{R}^n$  be an immersion, that gives a parametrization of the self-intersection manifold of the immersion  $g$ .

**Proposition 9.** *The normal  $4k$ -dimensional bundle  $\nu_L$  of the immersion  $h$  is isomorphic to the Whitney sum of  $k$  copies of a  $4$ -dimensional bundle  $\zeta$  over  $L^{n-4k}$ , each  $4$ -dimensional direct summand has the structure group  $\mathbb{Z}/2^{[3]}$  and is classified by a classifying mapping  $\zeta : L^{n-4k} \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$ .*

### Proof of Proposition 9

The proof is omitted, this proof is analogous to the proof of Proposition 5.

**Definition 10.** We define the cobordism group of immersions  $Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$ , assuming  $n > 4k$ . Let  $(h, \zeta, \Lambda)$  be a triple, which determines a  $\mathbb{Z}/2^{[3]}$ -framed immersion of codimension  $4k$ . Here  $h : L^{n-4k} \looparrowright \mathbb{R}^n$  is an immersion and  $\zeta : L^{n-4k} \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$  is the characteristic map of the  $\mathbb{Z}/2^{[3]}$ -framing  $\Lambda$ . The cobordism relation of triples is standard.

**Lemma 11.** *Under the assumption  $k_1 < k$ ,  $4k < n$ , the following commutative diagram of groups is well defined:*

$$\begin{array}{ccccc} Imm^{\mathbf{D}}(n-2k_1, 2k_1) & \xrightarrow{J^{\mathbf{D}}} & Imm^{\mathbf{D}}(n-2k, 2k) & \xrightarrow{h_k^{\mathbf{D}}} & \mathbb{Z}/2 \\ \downarrow \delta_{k_1}^{\mathbf{D}} & & \downarrow \delta_k^{\mathbf{D}} & & \parallel \\ Imm^{\mathbb{Z}/2^{[3]}}(n-4k_1, 4k_1) & \xrightarrow{J^{\mathbb{Z}/2^{[3]}}} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k) & \xrightarrow{h_k^{\mathbb{Z}/2^{[3]}}} & \mathbb{Z}/2. \end{array} \quad (26)$$

### Proof of Lemma 11

Let us define the homomorphisms in the diagram (26). The homomorphism  $h_k^{\mathbf{D}}$  is given by the characteristic number in the right side of the formula (18). The homomorphism  $h_k^{\mathbb{Z}/2^{[3]}}$  is defined by means of the characteristic number in the left side of the formula (24). The commutativity of the right square of the diagram is proved in Proposition 8.

We define the further homomorphisms in the diagram (26). The homomorphism

$$Imm^{\mathbb{Z}/2^{[3]}}(n - 4k_1, 4k_1) \xrightarrow{J^{\mathbb{Z}/2^{[3]}}} Imm^{\mathbb{Z}/2^{[3]}}(n - 4k, 4k)$$

is defined exactly as the homomorphism  $J^{\mathbf{D}}$  in the bottom row of the diagram (9). Namely, let a triple  $(h, \zeta, \Lambda)$  represent an element in the cobordism group  $Imm^{\mathbb{Z}/2^{[3]}}(n - 4k, 4k)$ , where  $h : L^{n-4k} \looparrowright \mathbb{R}^n$  is an immersion with  $\mathbb{Z}/2^{[3]}$ -framing  $\Lambda$ . Take the universal  $\mathbb{Z}/2^{[3]}$ -bundle  $\psi_{[3]}$  over  $K(\mathbb{Z}/2^{[3]}, 1)$  and take the pull-back of this bundle by means of the classifying map  $\zeta, \zeta^*(\psi_{[3]})$ . Take a submanifold  $L'^{n-4k} \subset L^{n-4k_1}$  which represents the Euler class of the bundle  $(k - k_1)\eta^*(\psi_{[3]})$ . The triple  $(h', \zeta', \Lambda')$  is well defined, where  $h' = h|_{L'}$  and  $\zeta' = \zeta|_{L'}$ . Let us define the  $\mathbb{Z}/2^{[3]}$ -framing  $\Lambda'$ .

Let us consider the normal bundle  $\nu_{h'}$  of the immersion  $h'$ . This bundle decomposes into the Whitney sum of the two bundles:  $\nu_{h'} = \nu_h|_L \oplus \nu_{L' \subset L}$ , where  $\nu_{L' \subset L}$  is the normal bundle of the submanifold  $L'^{n-2k_1} \subset L^{n-2k}$ . The bundles  $\nu_{L' \subset L}$  and  $(k - k_1)\zeta^*(\psi_{[3]})$  are isomorphic and this bundle is equipped with the standard  $\mathbb{Z}/2^{[3]}$ -framing. The bundle  $\nu_h|_L$  is also equipped with the  $\mathbb{Z}/2^{[3]}$ -framing. Therefore the bundle  $\nu_h|_L$  is equipped with the dihedral framing  $\Lambda' : \nu_{h'} \cong k_1\zeta^*(\psi_{[3]})$ . The triple  $(h', \zeta', \Lambda')$  represent the element  $J^{\mathbb{Z}/2^{[3]}}(h, \zeta, \Lambda) \in Imm^{\mathbb{Z}/2^{[3]}}(n - 4k_1, 4k_1)$ .

The homomorphism

$$Imm^{\mathbf{D}}(n - 2k, 2k) \xrightarrow{\delta_k^{\mathbf{D}}} Imm^{\mathbb{Z}/2^{[3]}}(n - 4k, 4k)$$

transforms the cobordism class of a triple  $(g, \eta, \Psi)$  to the cobordism class of the triple  $(h, \zeta, \Lambda)$ , where  $h : L^{n-4k} \looparrowright \mathbb{R}^n$  is the immersion parameterizing the self-intersection points manifold of the immersion  $g$  (it is assumed that the immersion  $g$  intersects itself transversely),  $\Lambda$  is the  $\mathbb{Z}/2^{[3]}$ -framing of the normal bundle of the immersion  $h$ , and  $\zeta$  is the classifying mapping of the  $\mathbb{Z}/2^{[3]}$ -framing  $\Lambda$ .

The commutativity of the left square in the diagram (26) is proved analogously with the commutativity of the left square in the diagram (9). Lemma 11 is proved.

## 2 Proof of the main theorem

We reformulate the Main Theorem ( $\sigma(n)$  is defined by the formula (1)), taking into account the notation of the previous section.

**Theorem 12.** *For  $\ell \geq 7$  the homomorphism  $h_{2^{\ell-2}-n_\sigma}^{sf} : Imm^{sf}(3 \cdot 2^{\ell-2} + n_{\sigma-4}, 2^{\ell-2} - 1 - n_{\sigma-4}) \rightarrow \mathbb{Z}/2$ , given by the equivalent Definitions 1 and 3 is trivial.*

**Remark 13.** In the cases  $\ell = 4$  the proof by means of the considered approach is unknown, because of the dimensional restriction in Lemma 28, in the case  $\ell = 5, 6, 7$  because of the dimensional restrictions in Lemma 31. For a possible approach in the cases  $n = 31$ ,  $n = 63$  and  $n = 127$  see a remark in the Introduction of the part III [A3].

Consider the homomorphism  $J^{sf} : Imm^{sf}(n-1, 1) \rightarrow Imm^{sf}(3 \cdot 2^{\ell-2} + n_{\sigma-4}, 2^{\ell-2} - 1 - n_{\sigma-4})$ . According to Proposition 2,  $h_1^{sf} = h_{2^{\ell-2}-n_\sigma-1}^{sf} \circ J^{sf}$ . Let an element of the group  $Imm^{sf}(n-1, 1)$  be represented by an immersion  $f : M^{n-1} \looparrowright \mathbb{R}^n$ ,  $w_1(M) = \kappa$ . The value  $h_k^{sf}(J^{sf}(f))$ , where  $k = 2^{\ell-2} - n_\sigma - 1$ , coincides with the characteristic number  $\langle \kappa^{n-1}; [M^{n-1}] \rangle$ . Applying Theorem 12, we conclude the proof of the Main Theorem.

For the proof of Theorem 12 we shall need the fundamental Definitions 14, 15, 19, whose formulation will require some preparation.

### Definition of the subgroups $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{D}$ , $\mathbf{I}_b \subset \mathbf{D}$

We denote by  $\mathbf{I}_a \subset \mathbf{D}$  the cyclic subgroup of order 4 and index 2, containing the nontrivial elements  $a, a^2, a^3 \in \mathbf{D}$  (i.e., generated by the plane rotation which exchanges the coordinate axes). We denote by  $\mathbf{I}_d \subset \mathbf{I}_a$  the subgroup of index 2 with nontrivial elements  $a^2$ . We denote by  $\mathbf{I}_b \subset \mathbf{D}$  the subgroup of index 2 with nontrivial elements  $a^2, ab, a^3b$  (i.e., generated by the reflections with respect to the bisectors of the coordinate axes).

The following inclusion homomorphisms of subgroups are well defined:  $i_{d,a} : \mathbf{I}_d \subset \mathbf{I}_a$ ,  $i_{d,b} : \mathbf{I}_d \subset \mathbf{I}_b$ . When the image coincides with the entire group  $\mathbf{D}$  the corresponding index for the inclusion homomorphism will be omitted:  $i_d : \mathbf{I}_d \subset \mathbf{D}$ ,  $i_a : \mathbf{I}_a \subset \mathbf{D}$ ,  $i_b : \mathbf{I}_b \subset \mathbf{D}$ .

**Definition of the subgroup  $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$**

Let  $\mathbf{Q}$  be the quaternion group of order 8. This group has presentation  $\{\mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}$ . There is a standard representation  $\chi : \mathbf{Q} \rightarrow O(4)$ . The representation  $\chi$  (a matrix acts to the left on a vector) carries the unit quaternions  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  to the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (27)$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (28)$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (29)$$

These matrices give the action of **left** multiplication by  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  on the standard basis for the quaternions. This representation  $\chi$  defines the subgroup  $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]} \subset O(4)$ .

**Definition of the subgroups  $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{Q}$**

Denote by  $i_{\mathbf{I}_d, \mathbf{Q}} : \mathbf{I}_d \subset \mathbf{Q}$  the central subgroup of the quaternion group, which is also the center of the whole group  $\mathbb{Z}/2^{[3]}$ .

Denote by  $i_{\mathbf{I}_a, \mathbf{Q}} : \mathbf{I}_a \subset \mathbf{Q}$  the subgroup of the quaternion group generated by the quaternion  $\mathbf{i}$ .

The following inclusions are well defined:  $i_{\mathbf{I}_d} : \mathbf{I}_d \subset \mathbb{Z}/2^{[3]}$ ,  $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$ .

**Definition 14.** We say that a classifying map  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  is cyclic if it can be factored as a composition of a map  $\mu_a : N^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$  and the inclusion  $i_a : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$ . We say that the mapping  $\mu_a$  determines a reduction of the classifying mapping  $\eta$ .

**Definition 15.** We say that a classifying map  $\zeta : L^{n-4k} \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$  is quaternionic if it can be factored as a composition of a map  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$  and the inclusion  $i_{\mathbf{Q}} : K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$ . We also say that the mapping  $\lambda$  determines a reduction of the classifying mapping  $\zeta$ .

We shall later require the construction of the Eilenberg-Mac Lane spaces  $K(\mathbf{I}_a, 1)$ ,  $K(\mathbf{Q}, 1)$  and a description of the finite dimensional skeleta of these spaces, which we now recall.

Consider the infinite dimensional sphere  $S^\infty$  (a contractible space), which it is convenient to define as a direct limit of an infinite sequence of inclusions of standard spheres of odd dimension,

$$S^\infty = \varinjlim (S^1 \subset S^3 \subset \dots \subset S^{2j-1} \subset S^{2j+1} \subset \dots).$$

Here  $S^{2n-1}$  is defined by the formula  $S^{2j-1} = \{(z_1, \dots, z_j) \in \mathbb{C}^j, |z_1|^2 + \dots + |z_j|^2 = 1\}$ . Let  $\mathbf{i}(z_1, \dots, z_j) = (\mathbf{i}z_1, \dots, \mathbf{i}z_j)$ .

Then the space  $S^{2j-1}/\mathbf{i}$ , which is called the  $(2j-1)$ -dimensional lens space over  $\mathbb{Z}/4$ , is the  $(2j-1)$ -dimensional skeleton of the space  $K(\mathbf{I}_a, 1)$ . The space  $S^\infty/\mathbf{i}$  itself is the Eilenberg-Mac Lane space  $K(\mathbf{I}_a, 1)$ . The cohomology ring of this space is well-known, see e.g. [A-M].

Let us define the Eilenberg-Mac Lane space  $K(\mathbf{Q}, 1)$ . Consider the infinite dimensional sphere  $S^\infty$ , which now it is convenient to define as a direct limit of an infinite sequence of inclusions of standard spheres of dimensions  $4j+3$ :

$$S^\infty = \varinjlim (S^3 \subset S^7 \subset \dots \subset S^{4j-1} \subset S^{4j+3} \subset \dots).$$

A coordinate action  $\mathbf{Q} \times (\mathbb{C}^2)^j \rightarrow (\mathbb{C}^2)^j$  is defined on each direct summand  $\mathbb{H} = \mathbb{C}^2$  in accordance with the formulas (27), (28), (29). Thus, the space  $S^{4j-1}/\mathbf{Q}$  is a  $(4j-1)$ -dimensional skeleton of the space  $S^\infty/\mathbf{Q}$  and this space is called the  $(4j-1)$ -dimensional lens space over  $\mathbf{Q}$ . The space  $S^\infty/\mathbf{Q}$  itself is the Eilenberg-Mac Lane space  $K(\mathbf{Q}, 1)$ . The cohomology ring of this space is well known, see [At] section 13.

Denote by  $\psi_+ : E(\psi_+) \rightarrow S^{4k-1}/\mathbf{i}$  the restriction of the universal  $SO(2)$ -bundle over  $K(\mathbb{Z}/4, 1)$  on the standard skeleton of the dimension  $4k-1$ . Denote by  $\psi_- : E(\psi_-) \rightarrow S^{4k-1}/\mathbf{i}$  the  $SO(2)$ -bundle, which is the bundle  $\psi_+$  with the opposite orientation of fibers. Denote by  $T_{\mathbf{Q}} : S^{4k-1}/\mathbf{i} \rightarrow S^{4k-1}/\mathbf{i}$  the free involution, which is constructed by means of the normal subgroup  $\mathbf{I}_a \subset \mathbf{Q}$  of the index 2.

**Lemma 16.** *The following isomorphism of  $SO(2)$ -bundles is well defined:*

$$\psi_- \cong T_{\mathbf{Q}}^*(\psi_+). \quad (30)$$

### Proof of Lemma 16

Consider the trivial 2-bundle  $\mathbb{C} \times S^{4k-1} \rightarrow S^{4k-1}$ . Define the action  $\mathbf{I}_a \times \mathbb{C} \times S^{4k-1} \rightarrow \mathbb{C} \times S^{4k-1}$  by the formula:  $(\mathbf{i}, x, y) \mapsto (\mathbf{i}x, \mathbf{i}y)$ ,  $\mathbf{i} \in \mathbf{I}_a$ ,  $x \in \mathbb{C}$ ,  $y \in S^{4k-1}$ . The quotient of the considered action is the total space of the bundle  $\psi_+$ , which will be denoted by  $E(\psi_+)$ . The following projection  $E(\psi_+) \rightarrow S^{4k-1}/\mathbf{i}$  is well-defined, this is the required  $SO(2)$ -bundle  $\psi_+$ . Let us define the bundle  $\psi_-$  by the analogous construction, using the action  $(\mathbf{i}, x, y) \mapsto (-\mathbf{i}x, \mathbf{i}y)$ . The bundles  $\psi_+$  and  $\psi_-$  are isomorphic as  $O(2)$ -bundles, the isomorphism is given by the formula:  $(x, y)/\mathbf{i} \mapsto (R(x), y)/\mathbf{i}$ , where  $R(x)$  is the reflection with respect to the line, which is determined by the first base vector of the real fiber.

Let us prove that the bundles  $T_{\mathbf{Q}}(\psi_+)$  and  $\psi_-$  are isomorphic. Define the automorphism of the trivial bundle  $\mathbb{C} \times S^{4k-1}$  (which is not the identity on the base) by the formula:  $(R(x), y) \mapsto (x, T_{\mathbf{Q}}(y))$ . Because  $T_{\mathbf{Q}}(\mathbf{i}y) = -\mathbf{i}T_{\mathbf{Q}}(y)$  and  $R(\mathbf{i}(x)) = -\mathbf{i}R(x)$ , we get  $(R(\mathbf{i}x), \mathbf{i}y) \mapsto (-R(\mathbf{i}(x)), -T_{\mathbf{Q}}(\mathbf{i}y))$ . Therefore, the quotient  $\sim \mathbf{I}_a$  is well defined and the required isomorphism (30) is well defined. Lemma 16 is proved.

### Definition of the characteristic number $h_{\mu_a, k}$

Let us assume  $n > 4k$  and let us assume that on the manifold  $N^{n-2k}$  of self-intersection points of a skew-framed immersion there is defined a map  $\mu_a : N^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ . Define the characteristic value  $h_{\mu_a, k}$  by the formula:

$$h_{\mu_a, k} = \langle \bar{e}_g \bar{\mu}_a^* x; [\bar{N}_a^{n-2k}] \rangle, \quad (31)$$

where  $\bar{\mu}_a : \bar{N}_a^{n-2k} \rightarrow K(\mathbf{I}_d, 1)$  is a double cover over the map  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ , induced by the cover  $K(\mathbf{I}_d, 1) \rightarrow K(\mathbf{I}_a, 1)$ ,  $x \in H^{n-4k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$  is the generator,  $\bar{e}_g \in H^k(\bar{N}_a^{n-2k}; \mathbb{Z}/2)$  is the image of the Euler class  $e_g \in H^k(N^{n-2k}; \mathbb{Z}/2)$  of the immersion  $g$  by means of the covering  $p_a : \bar{N}_a^{n-2k} \rightarrow N^{n-2k}$ ,  $\bar{e}_g = p_a^*(e_g)$ , and  $[\bar{N}_a^{n-2k}]$  is the fundamental class of the manifold  $\bar{N}_a^{n-2k}$ . (The manifold  $\bar{N}_a^{n-2k}$  coincides with the canonical 2-sheeted covering  $\bar{N}^{n-2k}$ , if and only if the classifying mapping  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$  is cyclic, see Definition 14.)

Let us assume that  $k \equiv 0 \pmod{2}$ . Then the characteristic number (31) is the reduction modulo 2 of the following characteristic number, denoted the same, determined modulo 4:

$$\langle e_g \mu_a^* x; [N^{n-2k}] \rangle, \quad (32)$$

where  $x \in H^{n-4k}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$  is the generator,  $e_g \in H^k(N^{n-2k}; \mathbb{Z}/4)$  is the Euler class of the co-oriented immersion  $g$  with coefficients modulo 4,  $[N^{n-2k}]$



is the fundamental class of the oriented manifold  $N^{n-2k}$  with coefficients modulo 4.

**Definition of the characteristic number  $h_{\lambda,k}$**

Let us assume  $n > 4k$  and let us assume that on the manifold  $L^{n-4k}$  of self-intersection points of a  $\mathbf{D}$ -framed immersion there is defined a map  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ . Define the characteristic value  $h_{\lambda}^k$  by the formula:

$$h_{\lambda,k} = \langle \bar{\lambda}^* y; [\bar{L}_{\mathbf{I}_d}] \rangle, \quad (33)$$

where  $y \in H^{n-4k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$  is a generator,

$$\bar{\lambda}_{\mathbf{I}_d} : \bar{L}_{\mathbf{I}_d}^{n-4k} \rightarrow K(\mathbf{I}_d, 1) \quad (34)$$

is a 4-sheeted cover over the map  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ , induced by the cover  $K(\mathbf{I}_d, 1) \rightarrow K(\mathbf{Q}, 1)$ , and  $[\bar{L}_{\mathbf{I}_d}]$  is the fundamental class of the manifold  $\bar{L}_{\mathbf{I}_d}^{n-4k}$ . (The manifold  $\bar{L}_{\mathbf{I}_d}^{n-4k}$  coincides with the canonical 4-sheeted covering  $\bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k}$ , if and only if the classifying mapping  $\zeta : L^{n-4k} \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$  is quaternionic, see Definition 15.)

Let us assume that  $k \equiv 0 \pmod{2}$ . Then the characteristic number (33) is the reduction modulo 2 of the following characteristic number, denoted the same, determined modulo 4:

$$\langle \bar{\lambda}^* y; [\bar{L}] \rangle, \quad (35)$$

where  $y \in H^{n-4k}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$  is the generator,  $[\bar{L}]$  is the fundamental class of the oriented manifold  $\bar{L}^{n-4k}$  (the manifold  $\bar{L}^{n-4k}$  is the canonical 2-sheeted covering over the manifold  $L^{n-4k}$ ) with coefficients modulo 4.

**Lemma 17.** *For an arbitrary skew-framed immersion  $(f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa, \Xi)$  with self-intersection manifold  $N^{n-2k}$  for which the classifying mapping  $\eta$  of the normal bundle is cyclic, the following equality is satisfied:*

$$h_k^{sf}(f, \kappa, \Xi) = h_{\mu_a, k},$$

where the characteristic value on the right side is calculated for a mapping  $\mu_a$ , satisfying the condition  $\eta = i_a \circ \mu_a$ ,  $i_a : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$ .

**Proof of Lemma 17**

Consider the double cover  $\bar{\mu}_a : \bar{N}_a^{n-2k} \rightarrow K(\mathbf{I}_d, 1)$  over the mapping  $\mu_a : N^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ , induced by the double cover  $K(\mathbf{I}_d, 1) \rightarrow K(\mathbf{I}_a, 1)$

over the target space of the map. Since the structure mapping  $\eta$  is cyclic, the manifold  $\bar{N}_a^{n-2k}$  coincides with the canonical 2-sheeted cover  $\bar{N}^{n-2k}$  over the self-intersections manifold  $N^{n-2k}$  of the immersion  $f$ , the class  $\bar{e}_g \in H^{2k}(N^{n-2k}; \mathbb{Z}/2)$  coincides with the class  $\bar{\eta}_{sf}^{2k}$ ,  $\bar{\eta}_{sf} \in H^1(\bar{N}^{n-2k}; \mathbb{Z}/2)$ . The proof of the lemma follows from Lemma 7 since the mappings  $\bar{\mu}_a$  and  $\bar{\eta}$  coincide and the characteristic number  $h_{\mu_a, k}$  is computed as in the right side of the equation (18).

**Lemma 18.** *For an arbitrary  $\mathbf{D}$ -framed immersion  $(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)$  with a self-intersection manifold  $L^{n-4k}$ , for which the classifying mapping  $\zeta$  of the normal bundle is quaternionic, the following equality is satisfied:*

$$h_k^{sf}(g, \eta, \Psi) = h_{\lambda, k},$$

where the characteristic value on the right side is calculated by the formula (33) for a mapping  $\lambda$ , satisfying the condition  $\zeta = i_a \circ \lambda$ ,  $i_a : K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$ .

### Proof of Lemma 18

Define the 2-sheeted covering

$$\bar{\lambda}_{\mathbf{I}_a} : \bar{L}_{\mathbf{I}_a}^{n-4k} \rightarrow K(\mathbf{I}_a, 1) \quad (36)$$

over the mapping  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ , induced by the 2-sheeted cover  $K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{Q}, 1)$  over the target space of the map. Let us consider the 4-sheeted covering  $\bar{\lambda}_{\mathbf{I}_d} : \bar{L}_{\mathbf{I}_d}^{n-4k} \rightarrow K(\mathbf{I}_d, 1)$  over the mapping  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ , defined by the formula (34), over the target space of the map.

Since the structure mapping  $\zeta$  is quaternionic, the manifold  $\bar{L}_{\mathbf{I}_d}^{n-4k}$  coincides with the canonical 4-sheeted covering  $\bar{L}^{n-4k}$  over the self-intersections manifold  $L^{n-4k}$  of the immersion  $g$ . The proof of the lemma follows from Proposition 8, since the mappings  $\bar{\lambda}$  and  $\bar{\zeta}$  coincide and the characteristic number  $h_{\lambda}^k$  is computed as in the left side of the equation (24).

Let us define the subgroup  $\mathbf{H}_b \subset \mathbb{Z}/2^{[3]}$  as a product of the subgroup  $\mathbf{I}_a \subset \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$  and an elementary subgroup, with the only non-trivial element  $t$  given by the transformation transposing each pair of the corresponding basis vectors  $\mathbf{e}_1 = \mathbf{1}$  and  $\mathbf{e}_3 = \mathbf{j}$  and the pair of the basis vectors  $\mathbf{e}_2 = \mathbf{i}$  and  $\mathbf{e}_4 = \mathbf{k}$ , preserving their direction. It is easy to verify that the group  $\mathbf{H}_b$  has the order 8 and this group is isomorphic to  $\mathbb{Z}/4 \times \mathbb{Z}/2$ . The groups  $\mathbf{H}_b$  and  $\mathbf{Q}$  contains the common index 2 subgroup:  $\mathbf{I}_a \subset \mathbf{H}_b$ ,  $\mathbf{I}_a \subset \mathbf{Q}$ .

**Definition 19.** Let  $(g, \Psi, \eta)$  be a  $\mathbf{D}$ -framed immersion, where the immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  is assumed to be in a general position with self-intersection manifold denoted by  $L^{n-4k}$ . Assume that the manifold  $N^{n-2k}$  contains a marked component  $N_a^{n-2k} \subset N^{n-2k}$ , with self-intersection manifold  $L_a^{n-4k} \subset L^{n-4k}$ .

Let the component  $N_a^{n-2k}$  be equipped with a mapping  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ , which is determined a reduction of the restriction of the classifying mapping  $\eta$  to the component  $N_a^{n-2k}$  (see property 1 in Definition (??)).

Assume that the manifold  $L_a^{n-4k}$  is the disjoint union of the two closed submanifolds:  $L_a^{n-4k} = L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_b}^{n-4k}$ . Moreover, there exists a pair of mappings  $(\mu_a, \lambda)$ , where  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ ,  $\lambda = \lambda_{\mathbf{Q}} \cup \lambda_{\mathbf{H}_b} : L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_b}^{n-4k} \rightarrow K(\mathbf{Q}, 1) \cup K(\mathbf{H}_b, 1)$ . Define the manifold  $\bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k}$  and its mapping

$$\bar{\lambda} = \bar{\lambda}_{\mathbf{Q}} \cup \bar{\lambda}_{\mathbf{H}_b} : \bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k} \rightarrow K(\mathbf{I}_a, 1) \cup K(\mathbf{I}_a, 1), \quad (37)$$

as the 2-sheeted covering mapping over the disjoint union of the mappings  $\lambda_{\mathbf{Q}} : L_{\mathbf{Q}}^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ ,  $\lambda_{\mathbf{H}_b} : L_{\mathbf{H}_b}^{n-4k} \rightarrow K(\mathbf{H}_b, 1)$  which is induced from 2-sheeted coverings  $K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{Q}, 1)$ ,  $K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{H}_b, 1)$  over the target space of the mapping  $\lambda$ . We say that this  $\mathbf{D}$ -framed immersion  $(g, \Psi, \eta)$  admits a quaternionic structure if the following two conditions are satisfied:

–1. The manifold  $\bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k}$  is diffeomorphic to the canonical 2-sheeted covering manifold  $\bar{L}^{n-4k}$  over the self-intersection manifold of the immersion  $g$ , and the mapping

$$(Id \cup Id) \circ \bar{\lambda} : \bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k} \rightarrow K(\mathbf{I}_a, 1) \cup K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{I}_a, 1), \quad (38)$$

where  $Id \cup Id : K(\mathbf{I}_a, 1) \cup K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{I}_a, 1)$  is the identity map over each component, coincides with the restriction of the mapping  $\mu_a$  to the submanifold  $\bar{L}^{n-4k} \looparrowright N^{n-2k}$ .

–2. The following equation is satisfied:

$$h_{\mu_a, k} = h_k^{\mathbf{D}}(g, \eta, \Xi), \quad (39)$$

where the characteristic number in the left side of the formula is given by (31) and on the right side is given by the formula (18).

**Example 20.** Let us assume that the classifying map  $\zeta$  is quaternionic. The quaternionic structure is defined by the mappings  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ ,  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$ , where  $i_{\mathbf{I}_a} \circ \mu_a = \eta$ ,  $i_{\mathbf{I}_a} : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$ ,  $i_{\mathbf{Q}} \circ \lambda = \zeta$ ,  $i_{\mathbf{Q}} : K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$ .

**Lemma 21.** *Assume that a pair  $(\mu_a, \lambda)$  determines a quaternionic structure for a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$ . Then the characteristic number  $h_{\lambda, k}$ , determined by the formula (33), coincides with the characteristic number  $h_{\mu_a}^k$ , given by the formula (31).*

### Proof of Lemma 21

The proof is analogous to the proof of Proposition 8.

**Corollary 22.** *Assume that a pair  $(\mu_a, \lambda)$  determines a quaternionic structure for a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$ . Then*

$$h_{\mu_a, k} = \langle \bar{e}_g \bar{\mu}_a^* x; [\bar{N}_a^{n-2k}] \rangle = h_{\lambda, k}(L_{\mathbf{Q}}) + h_{\lambda, k}(L_{\mathbf{H}_b}), \quad (40)$$

where the terms in the right side are defined by the formula (33) for each corresponding component of the manifold  $L^{n-4k}$ .

## 3 Cyclic structure for formal (equivariant) mappings

**Theorem 23.** *For  $k \geq 5$ , an arbitrary element of the group  $\text{Imm}^{\mathbf{D}}(n - 2k, 2k)$ , which is in the image of the homomorphism (10)  $J^{\mathbf{D}} : \text{Imm}^{sf}(n - k, k) \rightarrow \text{Imm}^{\mathbf{D}}(n - 2k, 2k)$ , is represented by a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$ , where  $g : N^{n-2k} \looparrowright \mathbb{R}^n$ ,  $N^{n-2k} = N_a^{n-2k} \cup N_{\mathbf{D}}^{n-2k}$ , the restriction of the classifying mapping  $\eta|_{N_a^{n-2k}}$  to the component  $N_a^{n-2k} \subset N^{n-2k}$  is cyclic in the sense of Definition 14, where the characteristic number (18), which is calculated for the mapping  $\eta|_{N_{\mathbf{D}}^{n-2k}}$ , is trivial.*

We need to reformulate the notion of a cyclic structure and of a quaternionic structure without the assumption that the corresponding maps  $f : M^{n-k} \rightarrow \mathbb{R}^n$  and  $g : N^{n-2k} \rightarrow \mathbb{R}^n$  are immersions. We formulate the necessary definition in minimal generality, under the assumption that  $M^{n-k} = \mathbb{RP}^{n-k}$ ,  $N^{n-2k} = S^{n-2k}/\mathbf{i}$ .

Let

$$d : \mathbb{RP}^{n-k} \rightarrow \mathbb{R}^n \quad (41)$$

be an arbitrary  $PL$ -mapping. Consider the two-point configuration space

$$(\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Delta_{\mathbb{RP}^{n-k}})/T', \quad (42)$$

which is called the “deleted square” of the space  $\mathbb{RP}^{n-k}$ . This space is obtained as the quotient of the direct product without the diagonal by the involution  $T' : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \rightarrow \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ , exchanging the coordinates. This space is an open manifold. It is convenient to define an analogous space, which is a manifold with boundary.

Define the space  $\bar{\Gamma}$  as a spherical blow-up of the space  $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag}$  in the neighborhood of the diagonal. The spherical blow-up is a manifold with boundary, which is defined as a result of compactification of the open manifold  $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag}$  by the fiberwise glue-in of the fibers of the unit sphere bundle  $ST\Sigma_{diag}$  of the tangent bundle  $T\Sigma_{diag}$  in the neighborhood of zero-sections of the normal bundle of the diagonal  $\Sigma_{diag} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ . The following natural inclusions are well defined:

$$\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag} \subset \bar{\Gamma},$$

$$ST\Sigma_{diag} \subset \bar{\Gamma}.$$

On the space  $\bar{\Gamma}$  the free involution  $\bar{T}' : \bar{\Gamma}_0 \rightarrow \bar{\Gamma}$ , which is an extension of the involution  $T'$  is well defined.

The quotient  $\bar{\Gamma}/\bar{T}'$  is denoted by  $\Gamma$ , and the corresponding double covering by

$$p_\Gamma : \bar{\Gamma}/\bar{T}' \rightarrow \Gamma.$$

The space  $\Gamma$  is a manifold with boundary and it is called the resolution space of the configuration space (42). The projection  $p_{\partial\Gamma} : \partial\Gamma \rightarrow \mathbb{RP}^{n-k}$  is well defined, and is called a resolution of the diagonal.

### Formal (equivariant) mapping with holonomic self-intersection

Denote by  $T_{\mathbb{RP}^{n-k}}, T_{\mathbb{R}^n}$  the standard involutions on the spaces  $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}, \mathbb{R}^n \times \mathbb{R}^n$ , which permutes the coordinates. Let

$$d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \rightarrow \mathbb{R}^n \times \mathbb{R}^n \quad (43)$$

be an arbitrary  $(T_{\mathbb{RP}^{n-k}}, T_{\mathbb{R}^n})$ -equivariant mapping, which is transversal along the diagonal of the source space. Denote  $(d^{(2)})^{-1}(\mathbb{R}_{diag}^n)/T_{\mathbb{RP}^{n-k}}$  by  $\mathbf{N} = \mathbf{N}(d^{(2)})$ , let us call this polyhedron a self-intersection (formal) polyhedron of the mapping  $d^{(2)}$ . In the case the formal mapping  $d^{(2)}$  is the extension

of a mapping (41), the polyhedron  $\mathbf{N}(d^{(2)})$  coincides with the polyhedron, denoted by the formula:

$$\mathbf{N}(d) = Cl\{([x, y]) \in \Gamma_\circ : y \neq x, d(y) = d(x)\}. \quad (44)$$

Note that  $int(\Gamma)$  is homeomorphic to  $\Gamma_\circ$ . Denote  $int(\mathbf{N}(d^{(2)})) = \mathbf{N}(d^{(2)}) \setminus (\mathbf{N}(d^{(2)}) \cap \Delta_\Gamma)$  by  $\mathbf{N}(d^{(2)})_\circ$ .

There is a canonical double covering

$$p_{\mathbf{N}} : \bar{\mathbf{N}} \rightarrow \mathbf{N}, \quad (45)$$

ramified over the boundary  $\partial\mathbf{N}$  (above this boundary the cover is a diffeomorphism). The following diagram is commutative:

$$\begin{array}{ccc} i_{\bar{\mathbf{N}}} : (\bar{\mathbf{N}}, \partial\bar{\mathbf{N}}) & \subset & (\bar{\Gamma}, \partial\Gamma) \\ \downarrow p_{\mathbf{N}} & & \downarrow p_\Gamma \\ i_{\mathbf{N}} : (\mathbf{N}, \partial\mathbf{N}) & \subset & (\Gamma, \partial\Gamma). \end{array}$$

**Structural map**  $\eta_{\mathbf{N}\circ} : \mathbf{N}(d)_\circ \rightarrow K(\mathbf{D}, 1)$

Define the mapping

$$\eta_\Gamma : \Gamma \rightarrow K(\mathbf{D}, 1), \quad (46)$$

which we shall call the structure mapping of the “deleted square”. Note that the inclusion  $\bar{\Gamma} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$  induces an isomorphism of fundamental groups, since the codimension of the diagonal  $\Delta_{\mathbb{RP}^{n-k}} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$  is equal to  $n - k$  and satisfies the inequality  $n - k \geq 3$ . Therefore, the equality is satisfied:

$$\pi_1(\bar{\Gamma}) = H_1(\bar{\Gamma}; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \quad (47)$$

Consider the induced automorphism  $T'_{ast} : H_1(\bar{\Gamma}; \mathbb{Z}/2) \rightarrow H_1(\bar{\Gamma}; \mathbb{Z}/2)$ . Note that this automorphism is not the identity. Fix an isomorphism of the groups  $H_1(\bar{\Gamma}; \mathbb{Z}/2)$  and  $\mathbf{I}_c$ , which maps the generator of the first (respectively second) factor of  $H_1(\bar{\Gamma}; \mathbb{Z}/2)$ , see (47), into the generator  $ab \in \mathbf{I}_c \subset \mathbf{D}$  (respectively,  $ba \in \mathbf{I}_c \subset \mathbf{D}$ ), which in the standard representation of the group  $\mathbf{D}$  is defined by the reflection with respect to the second (respectively, the first) coordinate axis.

It is easy to verify that the automorphism of the conjugation with respect to the subgroup  $\mathbf{I}_c \subset \mathbf{D}$  by means of the element  $b \in \mathbf{D} \setminus \mathbf{I}_c$  (in this formula

the element  $b$  can be chosen arbitrarily), defined by the formula  $x \mapsto bxb^{-1}$ , corresponds to the automorphism  $T'_*$ . The fundamental group  $\pi_1(\Gamma)$  is a quadratic extension of  $\pi_1(\bar{\Gamma})$  by means of the element  $b$ , and this extension is uniquely defined up to isomorphism by the automorphism  $T'_*$ . Therefore  $\pi_1(\Gamma) \simeq \mathbf{D}$ , and hence the mapping  $\eta_\Gamma : \Gamma \rightarrow K(\mathbf{D}, 1)$  is well defined.

It is easy to verify that the mapping  $\eta_\Gamma|_{\partial\Gamma}$  takes values in the subspace  $K(\mathbf{I}_{b \times b}, 1) \subset K(\mathbf{D}, 1)$ . The mapping  $\eta_\Gamma$ , which is defined by the formula (46), induces the mapping

$$\eta_{\mathbf{N}_\circ} : (\mathbf{N}_\circ, U(\partial\mathbf{N})_\circ) \rightarrow (K(\mathbf{D}, 1), K(\mathbf{I}_{b \times b}, 1)), \quad (48)$$

which we call the structure mapping.

Also, it is easy to verify that the homotopy class of the composition  $U(\partial\mathbf{N})_\circ \xrightarrow{\eta_{\mathbf{N}_\circ}} K(\mathbf{I}_{b \times b}, 1) \xrightarrow{p_b} K(\mathbf{I}_d, 1)$ , where the map  $K(\mathbf{I}_{b \times b}, 1) \xrightarrow{p_b} K(\mathbf{I}_d, 1)$  is induced by the homomorphism  $\mathbf{I}_{b \times b} \rightarrow \mathbf{I}_d$  with the kernel  $\mathbf{I}_b$ ,  $\partial\mathbf{N}(d) \xrightarrow{\eta} K(\mathbf{I}_b, 1) \xrightarrow{p_b} K(\mathbf{I}_d, 1)$  is extended to a map on  $\partial\mathbf{N}$  and this extension coincides to the map  $\kappa \circ res_d : \partial\mathbf{N}(d) \rightarrow \mathbb{R}P^{n-k} \rightarrow K(\mathbf{I}_d, 1)$ , which is the composition of the resolution map  $res_d : \partial\mathbf{N}(d) \rightarrow \mathbb{R}P^{n-k}$  and the embedding of the skeleton  $\mathbb{R}P^{n-k} \subset K(\mathbf{I}_d, 1)$  in the classifying space.

Let us assume that in the polyhedron  $\mathbf{N}_\circ$  there is a marked component

$$\mathbf{N}_a \subset \mathbf{N}_\circ \quad (49)$$

and assume that the following mapping

$$\mu_a : \mathbf{N}_a \rightarrow K(\mathbf{I}_a, 1) \quad (50)$$

determines the reduction of the restriction of the structure mapping (48) to the marked component in the formula (49).

The following characteristic number

$$\langle \mu_a^*(t); [\mathbf{N}_a] \rangle, \quad (51)$$

is well defined, where  $t \in H^{n-2k}(K(\mathbf{I}_a, 1); \mathbb{Z}/2)$  is the generic cohomology class  $[\mathbf{N}_a]$  is the fundamental class of the polyhedron  $\mathbf{N}_a$  (assuming that the equivariant  $PL$ -mapping  $d^{(2)}$  is transversal along  $\Delta_{\mathbb{R}^n}$ , this polyhedron is a  $PL$ -manifold).

**Definition 24. Cyclic structure of a formal mapping  $d^{(2)}$**

Assume that a component (49) of the polyhedron  $\mathbf{N}_\circ$  is marked and a mapping (50) is well defined. The mapping (50) is called the cyclic structure of the equivariant mapping with holonomic self-intersection along the polyhedron  $\mathbf{N}_a$ , if the characteristic number (51) satisfies the following equation:

$$\langle \mu_a^*(t); [\mathbf{N}_a] \rangle = 1. \quad (52)$$

We need a criterion to verify that the mapping  $\mu_a$  satisfies the equation (52).

Let us consider the canonical 2-sheeted covering over the polyhedron  $\mathbf{N}$ , which is, probably, branched over the boundary:

$$p : \bar{\mathbf{N}} \rightarrow \mathbf{N}. \quad (53)$$

The total space of this covering is a closed polyhedron  $\bar{\mathbf{N}}$  of the dimension  $n - 2k$ . This polyhedron is decomposed into the union of the two subpolyhedra:  $\bar{\mathbf{N}} = \bar{\mathbf{N}}_a \cup \bar{\mathbf{N}}_b$ , this decomposition corresponds with the decomposition in the formula (49). The polyhedron  $\bar{\mathbf{N}}_a$  is a closed  $PL$ -manifold of the dimension  $n - 2k$ , the polyhedron  $\bar{\mathbf{N}}_b$  is a compactification of an open  $PL$ -manifold of the dimension  $n - 2k$  by means the boundary  $\partial\mathbf{N}_b$ . The restriction of the involution  $T_p$  (53) on the boundary is free and a closed polyhedron  $\bar{\mathbf{N}}_b \cup_{T_p} T(\bar{\mathbf{N}}_b)$  is well-defined, denote this polyhedron by  $\mathbf{N}_b/\sim$ . Denote  $\bar{\mathbf{N}}_a \cup \mathbf{N}_b/\sim$  by  $\bar{\mathbf{N}}/\sim$ .

Let us consider the mapping  $p_{\mathbf{I}_c, \mathbf{I}_d} \circ \bar{\eta} : \bar{\mathbf{N}}/\sim \rightarrow K(\mathbf{I}_c, 1) \rightarrow K(\mathbf{I}_d, 1)$ , where the mapping  $p_{\mathbf{I}_c, \mathbf{I}_d} : K(\mathbf{I}_c, 1) \rightarrow K(\mathbf{I}_d, 1)$  is induced by the epimorphism  $\mathbf{I}_c \rightarrow \mathbf{I}_d$ , with the kernel, generated by the element  $ab \in \mathbf{I}_c$ . Denote this mapping by  $\bar{\eta} : \bar{\mathbf{N}}/\sim \rightarrow K(\mathbf{I}_d, 1)$ . Denote the restriction of  $\bar{\eta}$  on  $\bar{\mathbf{N}}_b/\sim \subset \bar{\mathbf{N}}/\sim$  by  $\bar{\eta}_b : \bar{\mathbf{N}}_b/\sim \rightarrow K(\mathbf{I}_d, 1)$ .

**Lemma 25.** *Assume that the restriction of the structure mapping  $\eta_o$  on the component  $\mathbf{N}_a$  as in Definition 24 and a mapping  $\mu_a$  determines a reduction of the structure mapping to the mapping into  $K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$ . The condition (51) is a corollary of the following two conditions: the homology class*

$$\bar{\eta}_{b*}([\bar{\mathbf{N}}_b/\sim]) \in H_{n-2k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2) \quad (54)$$

*is trivial.*

### Proof of Lemma 25

Let us consider a sketch of the proof. The homology class  $\bar{\eta}_*([\bar{\mathbf{N}}/\sim]) \in H_{n-2k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ , is the generator, because the fundamental class of the subpolyhedron  $\bar{\mathbf{N}}/\sim = Cl(\bar{\mathbf{N}}_o) \subset \mathbb{RP}^{n-k}$  is dual to the (normal) characteristic class of the dimension  $2k$ , which is the generator in  $H_{n-2k}(\mathbb{RP}^{n-k})$ , because  $n = 2^\ell - 1$ . Therefore the characteristic number (52) is equal to 1 iff the class (54) is trivial. Lemma 25 is proved.



Proposition 23 is based on the application of the following principle of density of the subspace of immersions in the space of continuous maps equipped with the compact-open topology, see [Hi, Theorem 5.10].

**Proposition 26.** *Let  $f_0 : M \looparrowright R$  (we will use the case  $R = \mathbb{R}^n$ ) be a smooth immersion between manifolds, where the manifold  $M$  is compact, the manifold  $R$  is equipped with the metric  $\text{dist}$  and  $\dim(M) < \dim(R)$ . Let  $g : M \rightarrow R$  be a continuous mapping homotopic to the immersion  $f_0$ . Then  $\forall \varepsilon > 0$  there exists an immersion  $f : M \looparrowright R$ , regularly homotopic to the immersion  $f_0$ , for which  $\text{dist}(g; f)_{C^0} < \varepsilon$  in the space of maps with the induced metric.*

**Proposition 27.** *Let  $(M^{(2)}, \partial M^{(2)})$  be a smooth manifold with boundary, assume that a free involution  $T_{M^{(2)}} : (M^{(2)}, \partial M^{(2)}) \rightarrow (M^{(2)}, \partial M^{(2)})$  is well defined. Let  $R^{(2)}$  be a smooth manifold with the metric, denoted by  $\text{dist}$ , equipped the standard involution  $T_{R^{(2)}} : R^{(2)} \rightarrow R^{(2)}$ , which is free outside the diagonal  $\Delta_R \subset R^{(2)}$  and  $\dim(R^{(2)}) = 2n$ ,  $\dim(\Delta_R) = n$ ,  $\dim(M^{(2)}) < \dim(R^{(2)})$ . Assume that there exists a  $(T_M, T_R)$ -equivariant immersion  $F_0^{(2)} : M^{(2)} \looparrowright R^{(2)}$ , and the image of the boundary has no intersection with the fixed point manifold:*

$$\text{Im}(F_0^{(2)}(\partial M^{(2)})) \subset R^{(2)} \setminus \Delta_R. \quad (55)$$

*Let  $G^{(2)} : M^{(2)} \rightarrow R^{(2)}$  be a continuous  $(T_{M^{(2)}}, T_{R^{(2)}})$ -equivariant mapping, for which the following condition is satisfied*

$$\text{Im}(G^{(2)}(\partial M^{(2)})) \subset R^{(2)} \setminus \Delta_R. \quad (56)$$

*Moreover, assume that the equivariant mapping  $G^{(2)}$  is equivariant homotopic to the immersion  $F_0^{(2)}$  in the space of mappings with the condition above. Then  $\forall \varepsilon > 0$  there exists a  $(T_{M^{(2)}}, T_{R^{(2)}})$ -equivariant immersion  $F_1^{(2)} : M^{(2)} \looparrowright R^{(2)}$ , which is regular equivariant homotopic to the equivariant immersion  $F_0^{(2)}$ , for which the analogous condition (55) is satisfied, and additionally, the condition  $\text{dist}(F_1^{(2)}; G^{(2)})_{C^0} < \varepsilon$  in the space of equivariant maps with induced metric.*

### A sketch of a proof of Proposition 27

Consider an equivariant triangulation of the manifold  $(M^{(2)}, \partial M^{(2)})$  of the caliber much less then  $\varepsilon$ . The proof is possible by the induction over the skeletons by analogous arguments as the Hirsch Theorem 26.

We shall use Proposition 27 as follows. Let  $(f, \kappa, \Psi)$  be a skew-framed immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$ . Consider an open manifold  $M^{n-k} \times M^{n-k} \setminus \Delta_M$ , which is equipped with the standard involution  $T_M$ , this involution is free outside the diagonal. Denote by  $\bar{M}^{(2)}$  the spherical blow-up of the manifold  $M^{n-k} \times M^{n-k} \setminus \Delta_M$ , equipped with the free involution  $T_M^{(2)}$ . Denote by  $M^{(2)}$  the quotient  $\bar{M}^{(2)}/T_M^{(2)}$ . The boundary  $\partial M^{(2)}$  coincides with the projectivization  $TP(M^{n-k})$  of the tangent bundle  $T(M^{n-k})$ . Denote the manifold  $M^{(2)} \setminus \partial M^{(2)}$  by  $M_o^{(2)}$ . Denote by  $(\mathbb{R}^n)^{(2)}$  the manifold  $\mathbb{R}^n \times \mathbb{R}^n$ , equipped with the standard involution  $T_{\mathbb{R}^n}^{(2)}$ . The following mapping of the classifying spaces is well defined:

$$\bar{f}^{(2)} : \bar{M}^{(2)} \rightarrow \mathbb{R}^n \times \mathbb{R}^n, \quad (57)$$

This mapping is  $(T_M^{(2)}, T_R^{(2)})$ -equivariant immersion. The equivariant immersion (57) satisfies the condition, which is analogous to the condition for the mapping  $F_0$  from Proposition 27.

Let us calculate the normal bundle  $\nu_{\bar{f}^{(2)}}$  of the immersion  $\bar{f}^{(2)}$ , using the skew-framing  $(\kappa, \Psi)$  of the immersion  $f$ . Evidently, the framing  $\Psi$  induces the isomorphism

$$\bar{\Psi}^{(2)} : \nu_{\bar{f}^{(2)}} = k(\kappa_1 \oplus \kappa_2), \quad (58)$$

where  $\kappa_i$  is the line bundle, which is induced by the immersion of the  $i$ -th factor,  $i = 1, 2$ . The involution  $T_M^{(2)}$  is covered by an involution of the bundle  $\nu_{\bar{f}^{(2)}}$  (this involution is not the identity over the bases of the bundles), which permutes the corresponding line factors in the right side of the formula (58). Therefore the vector bundle over  $M^{(2)}$  is well-defined, denote this bundle by  $\nu_{f^{(2)}}$ , the isomorphism (58) induces the isomorphism

$$\Psi^{(2)} : \nu_f^{(2)} = k(\eta), \quad (59)$$

where  $\eta$  is a 2-dimensional  $\mathbf{D}$ -bundle.

Let us assume that the mapping (57) is transversal along the diagonal  $\Delta_{\mathbb{R}^n} \subset \mathbb{R}^n \times \mathbb{R}^n$  and let us consider the inverse image  $(\bar{f}^{(2)})^{-1}(\Delta_{\mathbb{R}^n})$  of this diagonal, and denote this inverse image by  $\bar{N} \subset \bar{M}^{(2)}$ . Evidently, that  $\bar{N}$  is a closed  $n - 2k$ -dimensional manifold, which coincides with the manifold (4). The manifold  $\bar{N}$  is equipped with the standard free involution  $T_M^{(2)}|_{\bar{N}} : \bar{N} \rightarrow \bar{N}$ , denote the quotient with respect to this involution by  $N^{n-2k}$ . The manifold  $N^{n-2k}$  is closed and this manifold coincides with the manifold (3). This new definition of the manifold is more general, because this definition is possible without an assumption that the equivariant

immersion  $\bar{f}^{(2)}$  is holonomic. We shall use this definition for  $(T_M^{(2)}, T_R^{(2)})$ -equivariant immersions, which satisfy the condition (55) from the equivariant regular homotopy class of the equivariant immersion  $\bar{f}^{(2)}$ . The immersion  $\bar{N}^{n-2k} \looparrowright M^{n-k}$  and the restriction  $f|_{\bar{N}^{n-2k}} : \bar{N}^{n-2k} \looparrowright \mathbb{R}^n$  is well defined. The normal bundle of the immersion  $f|_{\bar{N}^{n-2k}}$  is isomorphic to the restriction of the bundle  $\nu_{\bar{f}^{(2)}}$  over the submanifold  $\bar{N} \subset \bar{M}^{(2)}$ .

Let us consider the immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$ , which is defined by the formula (5). The normal bundle  $\nu_g$  of this immersion  $g$  is naturally isomorphic to the restriction of the bundle  $\nu_f^{(2)}$  over the submanifold  $N^{n-2k} \subset M^{(2)}$ . The dihedral framing, which is determined by the formula (59), coincides with the dihedral framing  $\Psi$ , which is determined in Proposition 5.

Assume, that a  $(T_M^{(2)}, T_{\mathbb{R}^n}^{(2)})$ -equivariant immersion  $G^{(2)} : M^{(2)} \looparrowright \mathbb{R}^n \times \mathbb{R}^n$  is well defined, this immersion satisfies the condition (56), and is regular homotopic to the immersion (57) in the space of equivariant immersions with the prescribed conditions, in particular, regular homotopies keep the isomorphism (59). Assume that the regular homotopy keeps the condition (56) and is transversal to  $\Delta_{\mathbb{R}^n} \subset \mathbb{R}^n \times \mathbb{R}^n$ . Let us consider the manifold  $N^{n-2k}(G)$ . This manifold is close, additionally the immersion  $g(G) : N^{n-2k}(G) \looparrowright \Delta_{\mathbb{R}^n} = \mathbb{R}^n$  is well defined as the restriction of the immersion  $G$ . The normal bundle of the considered immersion, which is denoted by  $\nu_{g(G)}$ , is isomorphic to the restriction of the normal bundle of the immersion  $G/T_M$ . The normal bundle  $\nu_{g(G)}$  is equipped with a dihedral framing (59). Therefore a  $\mathbf{D}$ -framed immersion  $(g(G), \eta(G), \Psi(G))$  is regular cobordant to a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$  of the self-intersection manifold of the given skew-framed immersion  $(f, \kappa, \Xi)$ .

A key lemma for Theorem 23 is following, this lemma is proved in Part III of the paper.

**Lemma 28.** *Assuming the dimensional conditions*

$$k \geq 5, \quad n - k \equiv -1 \pmod{4}, \quad (60)$$

*in particular, for*

$$n = 2^\ell - 1, \quad \ell \geq 8, \quad n - 4k = 7, \quad (61)$$

*there exists a formal (equivariant) mapping  $d^{(2)}$ , see (43), which admits a cyclic structure in the sense of Definition 24.*

### Proof of Theorem 23 from Lemma 28

Consider the skew-framed immersion  $(f, \kappa, \Xi)$ , which is determined in the statement of Proposition 23. Consider the equivariant mapping  $d^{(2)} : \mathbb{R}P^{n-k} \times$

$\mathbb{RP}^{n-k} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ , which is constructed in Lemma 28. Consider the mapping  $\kappa : M^{n-k} \rightarrow \mathbb{RP}^N$ ,  $N \gg n-k$ . Without loss of a generality, we may assume that  $Im(\kappa) \subset \mathbb{RP}^{n-k} \subset \mathbb{RP}^\infty$ . Consider the mapping  $\kappa \times \kappa : M^{n-k} \times M^{n-k} \rightarrow \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ . Denote by  $(T_M, T_{\mathbb{RP}^{n-k}})$  the standard involutions in the image and in the preimage of the mapping  $\kappa \times \kappa$ . Denote the  $(T_M, T_{\mathbb{RP}^{n-k}})$ -equivariant mapping  $\kappa \times \kappa$  by  $\kappa^{(2)}$ . Let us consider the composition  $d^{(2)} \circ \kappa^{(2)} : M^{n-k} \times M^{n-k} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ , where the equivariant mapping  $d^{(2)}$  is constructed in Lemma 28.

Let us consider the restriction of the mapping  $d^{(2)} \circ \kappa^{(2)}$  on the diagonal  $\Delta_M \rightarrow \Delta_{\mathbb{R}^n}$ . By Proposition 26 the mapping  $d^{(2)} \circ \kappa^{(2)}$  is arbitrary closed to an equivariant immersion from the equivariant regular homotopy class of the equivariant immersion  $f^{(2)}$ . Therefore there exists an equivariant mapping  $G^{(2)} : (M^{n-k})^{(2)} \rightarrow (\mathbb{R}^n)^{(2)}$ , such that the restriction of this mapping on the interior of  $M^{n-k})_\circ^{(2)} \subset M^{n-k})^{(2)}$  is arbitrary closed to the restriction  $d^{(2)} \circ \kappa^{(2)}$  on  $M^{n-k})_\circ^{(2)} = M^{n-k} \times M^{n-k} \setminus \Delta_{M^{n-k}}$ , and which satisfies the condition (56). By Proposition 27 the equivariant immersion with the prescribed properties exists.

Consider the marked closed component  $\mathbf{N}_a$  of the polyhedron of the formal self-intersection of the formal mapping  $d^{(2)}$ . The closed component  $N'_a \subset (M^{n-k})^{(2)}$ , which is mapped to the component  $\mathbf{N}_a$  by the mapping  $\kappa^{(2)}$  is well-defined. Therefore, if  $\varepsilon > 0$  is sufficiently small, the immersion  $G^{(2)}$  contains the component  $N_a^{n-2k}(G^{(2)})$ , which is projected on  $\mathbf{N}_a$ , and the degree of this projection is equal to  $\deg^2(\kappa) = h_k \pmod{2}$ , where the characteristic number  $h_k$  is given by Definition 1 for  $(f, \kappa, \Xi)$ . The manifold  $N_a^{n-2k}$  is closed. The restriction of the framing  $\mathbf{D}$ -framing  $(g, \eta, \Psi)$  on the considered component is well-defined, in particular, the restriction of the characteristic mapping  $\eta$  to the component  $N_a^{n-2k}(G^{(2)}) \subset N^{n-2k}(G^{(2)})$  is well-defined. This restriction admits a reduction mapping  $\eta_a : N_a^{n-2k}(G^{(2)}) \rightarrow K(\mathbf{I}_a, 1)$ , which is induced from the cyclic mapping of the cyclic structure of the mapping  $d^{(2)}$ . The manifold  $N^{n-2k}(G^{(2)})$  is equipped with a  $\mathbf{D}$ -framed immersion  $(g(G^{(2)}), \eta(G^{(2)}), \Psi(G^{(2)}))$ , this  $\mathbf{D}$ -framed immersion represents the prescribed element  $[(g, \eta, \Psi)]$  in  $Imm^{\mathbf{D}}(n-2k, 2k)$ . Theorem 23 is proved.

## 4 Quaternionic structure of $PL$ -mappings with singularities

Let  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  be a  $PL$ -mapping in a general position.

Consider the configuration space

$$((S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Delta_{S^{n-2k}})/T', \quad (62)$$

which is called the “deleted square” of the lens space  $S^{n-2k}/\mathbf{i}$ . This space is obtained as the quotient of the direct product without the diagonal by the involution  $T' : S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i} \rightarrow S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$ , exchanging the coordinates. This space is an open manifold. It is convenient to define an analogous space, which is a manifold with boundary.

Define the space  $\bar{\Gamma}_1$  as a spherical blow-up of the space  $(S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Delta_{S^{n-2k}}$  in the neighborhood of the diagonal. The spherical blow-up is a manifold with boundary, which is defined as a result of compactification of the open manifold  $(S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Sigma_{diag}$  by the fiberwise glue-in of the fibers of the spherization  $ST\Sigma_{diag}$  of the tangent bundle  $T\Sigma_{diag}$  in the neighborhood of zero-sections of the normal bundle of the diagonal  $\Sigma_{diag} \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$ . The following natural inclusions are well defined:

$$S^{n-2k} \times S^{n-2k} \setminus \Sigma_{diag} \subset \bar{\Gamma}_1,$$

$$ST\Sigma_{diag} \subset \bar{\Gamma}_1.$$

On the space  $\bar{\Gamma}_1$  the free involution

$$\bar{T}' : \bar{\Gamma}_1 \rightarrow \bar{\Gamma}_1 \quad (63)$$

which is an extension of an involution  $T'$  is well defined.

The quotient  $\bar{\Gamma}_1/\bar{T}'$  is denoted by  $\Gamma_1$ , and the corresponding double covering by

$$p_{\Gamma_1} : \bar{\Gamma}_1/\bar{T}' \rightarrow \Gamma_1.$$

The space  $\Gamma_1$  is a manifold with boundary and it is called the resolution space of the configuration space (62). The projection  $p_{\partial\Gamma_1} : \partial\Gamma_1 \rightarrow S^{n-2k}/\mathbf{i}$  is well defined, this map is called a resolution of the diagonal.

For an arbitrary mapping  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  the polyhedron  $\mathbf{L}(c)$  of self-intersection points of the mapping  $c$  is defined by the formula:

$$\mathbf{L}(c) = Cl\{([x, y]) \in int(\Gamma_1) : y \neq x, c(y) = c(x)\}. \quad (64)$$

By Porteous' Theorem [Por] under the assumption that the map  $c$  is smooth and generic, the polyhedron  $\mathbf{L}(c)$  is a manifold with boundary of dimension  $n - 4k$ . This polyhedron is denoted by  $\mathbf{L}^{n-4k}(c)$  and called the polyhedron of self-intersection of the map  $c$ . This formula (64) defines an embedding of polyhedra into manifold:

$$i_{\mathbf{L}(c)} : (\mathbf{L}^{n-4k}(c), \partial\mathbf{L}^{n-4k}(c)) \subset (\Gamma_1, \partial\Gamma_1).$$

The boundary  $\partial\mathbf{L}^{n-4k}(c)$  of the manifold  $\mathbf{L}^{n-4k}(c)$  is called the resolution manifold of critical points of the map  $c$ . The map  $p_{\partial\Gamma_1} \circ i_{\partial\mathbf{L}(c)}|_{\partial\mathbf{L}(c)} :$

$\partial \mathbf{L}^{n-4k}(c) \subset \partial \bar{\Gamma}_1 \rightarrow S^{n-2k}$  is called the resolution map of singularities of the map  $c$ , we denote this mapping by  $res_c : \partial \mathbf{L}(c) \rightarrow S^{n-2k}/\mathbf{i}$ .

Consider the canonical double covering

$$p_{\mathbf{L}(c)} : \bar{\mathbf{L}}(c)^{n-4k} \rightarrow \mathbf{L}(c)^{n-4k}, \quad (65)$$

ramified over the boundary  $\partial \mathbf{L}(c)^{n-4k}$  (over this boundary the cover is a diffeomorphism). The next diagram is commutative:

$$\begin{array}{ccc} i_{\bar{\mathbf{L}}(c)} : (\bar{\mathbf{L}}^{n-4k}(c), \partial \mathbf{L}^{n-4k}(c)) & \subset & (\bar{\Gamma}_1, \partial \Gamma_1) \\ \downarrow p_{\mathbf{L}(c)} & & \downarrow p_{\Gamma_1} \\ i_{\mathbf{L}(c)} : (\mathbf{L}^{n-4k}(c), \partial \mathbf{L}^{n-4k}(c)) & \subset & (\Gamma_1, \partial \Gamma_1). \end{array}$$

**Definition of the subgroup  $\mathbf{H} \subset \mathbb{Z}/2^{[3]}$**

Consider the space  $\mathbb{R}^4$  with the basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ . The basis vectors are conveniently identified with the basic unit quaternions  $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ , which sometimes will be used to simplify the formulas for some transformations. Define the subgroup

$$\mathbf{H} \subset \mathbb{Z}/2^{[3]} \quad (66)$$

as a subgroup of transformations, of the following two types:

- in each plane  $Lin(\mathbf{e}_1 = \mathbf{1}, \mathbf{e}_2 = \mathbf{i})$ ,  $Lin(\mathbf{e}_3 = \mathbf{j}, \mathbf{e}_4 = \mathbf{k})$  may be (mutually independent) transformations by the multiplication with the quaternion  $\mathbf{i}$ . The subgroup of all such transformations is denoted by  $\mathbf{H}_c$ , this subgroup is isomorphic to  $\mathbb{Z}/4 \times \mathbb{Z}/4$ .

- the transformation exchanging each pair of the corresponding basis vectors  $\mathbf{e}_1 = \mathbf{1}$  and  $\mathbf{e}_3 = \mathbf{j}$  and the pair of the basis vectors  $\mathbf{e}_2 = \mathbf{i}$  and  $\mathbf{e}_4 = \mathbf{k}$ , preserving their direction. Denote this transformation by

$$t \in \mathbf{H} \setminus \mathbf{H}_c. \quad (67)$$

It is easy to verify that the group itself is  $\mathbf{H}$ , has order 32, and is a subgroup  $\mathbf{H} \subset \mathbb{Z}/2^{[3]}$  of index 4.

**Definition of the subgroup  $\mathbf{H}_{b \times b} \subset \mathbf{H}$  and the monomorphism  $i_{\mathbf{I}_a, \mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$**

Define the inclusion  $i_{\mathbf{I}_a, \mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$ , which translates the generator of the group  $\mathbf{I}_a$  into the operator of multiplication by the quaternion  $\mathbf{i}$ , acting simultaneously in each plane  $Lin(\mathbf{e}_1 = \mathbf{1}, \mathbf{e}_2 = \mathbf{i})$ ,  $Lin(\mathbf{e}_3 = \mathbf{j}, \mathbf{e}_4 = \mathbf{k})$ . Define

the subgroup  $\mathbf{H}_{b \times b} \subset \mathbf{H}$  as the product of the subgroup  $i_{\mathbf{I}_a, \mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$  and the subgroup generated by the generator  $t \in \mathbf{H}$ . It is easy to verify that the group  $\mathbf{H}_{b \times b}$  is of the order 8 and this group is isomorphic to  $\mathbb{Z}/4 \times \mathbb{Z}/2$ . The subgroup  $\mathbf{I}_a \subset \mathbf{H}_{b \times b}$  is of the index 2. The inclusion homomorphism  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}} : \mathbf{I}_a \subset \mathbf{H}_{b \times b}$  of the subgroup and the projection  $p_{\mathbf{H}_{b \times b}, \mathbf{I}_a} : \mathbf{H}_{b \times b} \rightarrow \mathbf{I}_a$  are defined, such that the composition  $\mathbf{I}_a \xrightarrow{i_{\mathbf{I}_a, \mathbf{H}_{b \times b}}} \mathbf{H}_{b \times b} \xrightarrow{p_{\mathbf{H}_{b \times b}, \mathbf{I}_a}} \mathbf{I}_a$  is the identity.

**Structure map**  $\zeta_\circ : \mathbf{L}_\circ(c) \rightarrow K(\mathbf{H}, 1)$

Define the map  $\zeta_{\Gamma_1} : \Gamma_1 \rightarrow K(\mathbf{H}, 1)$ , which we call the structure mapping of the “deleted square”. Note that the inclusion  $\bar{\Gamma}_1 \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$  induces an isomorphism of fundamental groups, since the codimension of the diagonal  $\Delta_{S^{n-2k}/\mathbf{i}} \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$  satisfies the inequality  $n - 2k \geq 3$ . Therefore, the following equality is satisfied:

$$\pi_1(\bar{\Gamma}_1) = H_1(\bar{\Gamma}_1; \mathbb{Z}/4) = \mathbb{Z}/4 \times \mathbb{Z}/4. \quad (68)$$

Consider the induced automorphism  $\bar{T}'_* : H_1(\bar{\Gamma}_1; \mathbb{Z}/2) \rightarrow H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$ , induced by the involution (63). Note that this automorphism is not the identity and permutes the factors. Fix an isomorphism of the groups  $H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$  and  $\mathbf{H}_c$ , which maps the generator of the first (respectively second) factor of  $H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$ , see (68), into the generator, defined by the multiplication by the quaternion  $\mathbf{i}$  in the plane  $\text{Lin}(\mathbf{1}, \mathbf{i})$  (respectively, in the plane  $\text{Lin}(\mathbf{j}, \mathbf{k})$ ) and is the identity on the complement.

It is easy to verify that the automorphism of the conjugation with respect to the subgroup  $\mathbf{H}_c \subset \mathbf{H}$  by means of the element  $t \in \mathbf{H} \setminus \mathbf{H}_c$ , (in this formula the element  $t$  is given by the equation (67)), defined by the formula  $x \mapsto txt^{-1}$ , corresponds to conjugation by means of the automorphism  $\bar{T}'_*$ . The fundamental group  $\pi_1(\Gamma_1)$  is a quadratic extension of  $\pi_1(\bar{\Gamma}_1)$  by means of the element  $t$ , and this extension is uniquely defined up to isomorphism by the automorphism  $T'_*$ . Therefore  $\pi_1(\Gamma_1) \simeq \mathbf{H}$ , and hence the mapping  $\zeta_{\Gamma_1} : \Gamma_1 \rightarrow K(\mathbf{H}, 1)$  is well defined.

It is easy to verify that the mapping  $\zeta_{\Gamma_1}|_{\partial\Gamma_1}$  takes values in the subspace  $K(\mathbf{H}_{b \times b}, 1) \subset K(\mathbf{H}, 1)$ . The mapping  $\zeta_{\Gamma_1}$  induces the map  $\zeta_\circ : (\mathbf{L}_\circ(c), U(\partial\mathbf{L}(c))_\circ) \rightarrow (K(\mathbf{H}, 1), K(\mathbf{H}_{b \times b}, 1))$ , which we call the structure map. (In the considered case the notion of the structure mapping is analogous to the notion of the classifying mapping for  $\mathbb{Z}/2^{[3]}$ -framed immersion.) Also, it is easy to verify that the homotopy class of the composition

$U(\partial \mathbf{L}(c)_\circ) \xrightarrow{\zeta_\circ} K(\mathbf{H}_{b \times \dot{b}}, 1) \xrightarrow{p_{\mathbf{H}_{b \times \dot{b}}, \mathbf{I}_a}} K(\mathbf{I}_a, 1)$  is extended to  $\partial \mathbf{L}(c)$  and coincides with the characteristic map  $\eta \circ res_c : \partial \mathbf{L}(c) \rightarrow S^{n-2k}/\mathbf{i} \rightarrow K(\mathbf{I}_a, 1)$ , which is the composition of the resolution map  $res_c : \partial \mathbf{L}(c) \rightarrow S^{n-2k}/\mathbf{i}$  and the embedding of the skeleton  $S^{n-2k}/\mathbf{i} \subset K(\mathbf{I}_a, 1)$  in the classifying space.

**Definition 29. Quaternionic structure for a mapping  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  with singularities**

Let  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  be a map in general position, having critical points, where  $k \equiv 0 \pmod{2}$ . Let  $\mathbf{L}(c)_\circ$  be the polyhedron of double self-intersection points of the map  $c$  with the boundary  $\partial \mathbf{L}(c)$ .

Let us assume that the polyhedron  $\mathbf{L}(c)_\circ$  is the disjoint union of the two components

$$\mathbf{L}(c)_\circ = \mathbf{L}_\mathbf{Q} \cup \mathbf{L}_{\mathbf{H}_{b \times \dot{b}} \circ}, \quad (69)$$

where the polyhedron  $\mathbf{L}_\mathbf{Q}$  is closed, and the polyhedron  $\mathbf{L}_{\mathbf{H}_{b \times \dot{b}} \circ}$ , generally speaking, contains a regular neighborhood of the boundary  $U(\partial \mathbf{L}_{\mathbf{H}_{b \times \dot{b}} \circ})_\circ = U(\partial \mathbf{L}(c))_\circ$ .

We say that this map  $c$  admits a (relative) quaternionic structure, if the structure map  $\zeta_\circ : (\mathbf{L}(c)_\circ, U(\partial \mathbf{L}(c))_\circ) \rightarrow (K(\mathbf{H}, 1), K(\mathbf{H}_{b \times \dot{b}}, 1))$  is given by the composition:

$$\begin{aligned} \lambda_{\mathbf{L}(c)} : \mathbf{L}_\mathbf{Q} \cup \mathbf{L}_{\mathbf{H}_{b \times \dot{b}} \circ} &\xrightarrow{\lambda_{\mathbf{L}_\mathbf{Q}} \cup \lambda_{\mathbf{H}_{b \times \dot{b}} \circ}} K(\mathbf{Q}, 1) \cup K(\mathbf{H}_{b \times \dot{b}}, 1) \\ &\xrightarrow{i_{\mathbf{Q}, \mathbf{H}} \cup i_{\mathbf{H}_{b \times \dot{b}}, \mathbf{H}}} K(\mathbf{H}, 1) \cup K(\mathbf{H}, 1) \xrightarrow{Id \cup Id} K(\mathbf{H}, 1). \end{aligned} \quad (70)$$

**Proposition 30.** *If  $n = 4k + n_\sigma$ ,  $n \geq 255$ , an arbitrary element of the group  $Imm^\mathbf{D}(n-2k, 2k)$ , in the image of the homomorphism  $\delta_k : Imm^{sf}(n-k, k) \rightarrow Imm^\mathbf{D}(n-2k, 2k)$ , is represented by a  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$ , admitting a quaternionic structure in the sense of Definition 19.*

**Lemma 31.** *Assuming  $n = 4k + n_\sigma$ ,  $n \geq 255$ , there exists a generic  $PL$ -mapping  $: S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  (with singularities), which admits a quaternionic structure in the sense of Definition 29.*

Proof of Lemma 31 is easily then the proof of Lemma 28 because of dimensional restrictions (see Part III of the paper).



### Proof of Proposition 30 from Proposition 23 and Lemma 31

Consider a skew-framed immersion  $(f, \kappa, \Xi)$ , such that  $\delta([(f, \kappa, \Xi)]) = [(g_1 : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$  represents the prescribed element in the group  $Imm^{\mathbf{D}}(n - 2k, 2k)$ . By Proposition 23, without loss of the generality, we may assume that the immersion  $f : M^{n-k} \looparrowright \mathbb{R}^n$  admits a closed marked component  $g : N_a^{n-2k} \looparrowright \mathbb{R}^n$ , equipped with the reduction mapping  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ .

Let us consider the mapping  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$ , which is constructed in Lemma 31 and consider the immersion  $g_a : N_a^{n-2k} \looparrowright \mathbb{R}^n$ , which is defined as a  $C^0$ -small approximation of the composition  $c \circ \mu_a : N_a^{n-2k} \rightarrow S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$  in the prescribed regular homotopy class of the restriction of the immersion  $g$ .

Let us consider the formal (holonomic) mapping of 2-configuration spaces. Analogously to Theorem 23 a quaternionic structure of the  $\mathbf{D}$ -framed immersion  $[(g, \eta, \Psi)]$  is well defined, this structure is induced from the quaternionic structure of the mapping  $c$ . Proposition 30 is proved.

## 5 Proof of Theorem 12

Let us take a positive integer  $k$  under the condition  $n - 4k = n_\sigma$ ,  $k \geq 8$ , this is possible if  $n \geq 127$ . Let the triple  $[(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$  represent the given element in the cobordism group  $Imm^{\mathbf{D}}(n - 2k, 2k)$ . Let us denote by  $L_a^{n-4k}$  the self-intersection manifold of the immersion  $g_a$ , which is the restriction of  $g$  on the marked component  $N_a^{n-2k} \subset N^{n-2k}$ . Let us consider a skew-framed immersion  $(f, \kappa, \Xi)$ , such that  $\delta_k^{sf}([(f, \kappa, \Xi)]) = [(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$ . By Proposition 30 we may assume that the triple  $[(g, \eta, \Psi)]$  admits a quaternionic structure in the sense of Definition 19.

In the first step let us assume that the classifying map  $\eta$  of the  $\mathbf{D}$ -framed immersion is cyclic in the sense of Definition 14. This means that for the marked component the following equation is satisfied:

$$\eta = i_a \circ \mu_a,$$

where  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ ,  $N_a^{n-2k} = N^{n-2k}$  and  $i_a : K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{D}, 1)$  is the natural map induced by the inclusion of the subgroup.

Let us also assume that the classifying map  $\zeta$  of the  $\mathbb{Z}/2^{[3]}$ -framed immersion  $\delta_k^{\mathbf{D}}(g, \eta, \Psi) = (h, \zeta, \Lambda)$  is quaternionic in the sense of Definition 15. This means that  $L^{n-4k} = L_{\mathbf{Q}}^{n-4k}$  and the following equation is satisfied:

$$\zeta = i_{\mathbf{Q}, \mathbb{Z}/2^{[3]}} \circ \lambda_{\mathbf{Q}}, \quad (71)$$

where  $\lambda_{\mathbf{Q}} : L_{\mathbf{Q}}^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ ,  $L_{\mathbf{Q}}^{n-4k} = L^{n-4k}$ , and  $i_{\mathbf{Q}, \mathbb{Z}/2^{[3]}} : K(\mathbf{Q}, 1) \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$  is the natural map, induced by the inclusion of the subgroup (see Example 20). Let us prove the theorem in this case.

Let us consider the classifying mapping  $\eta : N^{n-2k} \rightarrow K(\mathbf{D}, 1)$ . Let us denote by  $\tilde{N}^{n-2k-2} \subset N^{n-2k}$  the submanifold, representing the Euler class of the vector bundle  $\eta^*(\psi_{\mathbf{D}})$ , where by  $\psi_{\mathbf{D}}$  is denoted the universal 2-dimensional vector bundle over the classifying space  $K(\mathbf{D}, 1)$ . Because the classifying map  $\eta$  is cyclic, the submanifold  $\tilde{N}^{n-2k-2} \subset N^{n-2k}$  is co-oriented, moreover we have

$$\eta^*(\psi_{\mathbf{D}}) = \mu_a^*(\psi_+),$$

where by  $\psi_+$  we denote the 2-dimensional universal  $SO(2)$ -bundle over  $K(\mathbf{I}_a, 1)$ .

Let us denote by  $\tilde{g} : \tilde{N}^{n-2k-2} \hookrightarrow \mathbb{R}^n$  the restriction of the immersion  $g$  on the submanifold  $\tilde{N}^{n-2k-2} \subset N^{n-2k}$ , assuming that the immersion  $\tilde{g}$  is generic. The immersion  $\tilde{g}$  is a  $\mathbf{D}$ -framed immersion by  $\tilde{\Psi}$ , the classifying map  $\tilde{\eta}$  of this  $\mathbf{D}$ -framed immersion is the restriction of  $\eta$  to the submanifold, this map is cyclic. The triple  $(\tilde{g}, \tilde{\eta}, \tilde{\Psi})$  is constructed from the triple  $(g, \eta, \Psi)$  by means of the transfer homomorphism  $J^{\mathbf{D}}$  in the bottom row of the diagram (9) (in this diagram  $k_1$  is changed to  $k$ ,  $k$  is changed to  $k+1$ ).

Let us denote by  $\tilde{L}^{n-4k-4}$  the self-intersection manifold of the immersion  $\tilde{g}$ . The manifold  $\tilde{L}^{n-4k-4}$  is a submanifold of the manifold  $L^{n-4k}$ ,  $\tilde{L}^{n-4k-4} \subset L^{n-4k}$ . The parameterized immersion  $\tilde{h} : \tilde{L}^{n-4k-4} \hookrightarrow \mathbb{R}^n$  is well defined, this immersion is a  $\mathbb{Z}/2^{[3]}$ -framed immersion by means of  $\tilde{\Lambda}$ , the classifying map  $\tilde{\zeta}$  of this  $\mathbb{Z}/2^{[3]}$ -framed immersion is quaternionic. The triple  $(\tilde{h}, \tilde{\zeta}, \tilde{\Lambda})$  is defined from the triple  $(h, \zeta, \Lambda)$  by means of the homomorphism  $J^{\mathbb{Z}/2^{[3]}}$  in the bottom row of the diagram (26) (in this diagram  $k_1$  is changed to  $k$ ,  $k$  is changed to  $k+1$ ).

By Lemma 11 the submanifold  $\tilde{L}^{n-4k-4} \subset L^{n-4k}$  represents the Euler class of the bundle  $\zeta^*(\psi_{[3]})$ . This submanifold is the source manifold of a  $\mathbb{Z}/2^{[3]}$ -immersion, representing the image of the left bottom horizontal homomorphism in the diagram (26) (in the diagram  $k_1 = k$ ,  $k = k+1$ ).

Let us consider the canonical 2-sheeted covering  $\tilde{p} : \tilde{\tilde{L}}^{4k-4} \rightarrow \tilde{L}^{n-4k}$ . The submanifold  $\tilde{\tilde{L}}^{4k-4} \subset \tilde{L}^{n-4k}$  represents the Euler class of the bundle  $\tilde{p}^*(\zeta^*(\psi_{[3]}))$ . This vector bundle is naturally isomorphic to the vector bundle  $\tilde{\zeta}^*(\psi_{[3]}^!)$ , where  $\tilde{\zeta} : \tilde{L}^{n-4k} \rightarrow K(\mathbf{H}_c, 1)$  is the canonical 2-sheeted covering over the classifying map  $\zeta$  ( $\mathbf{H}_c \cong \mathbf{D} \times \mathbf{D}$ ),  $\psi_{[3]}^!$  is the pull-back of the universal vector bundle  $\psi_{[3]}$  over  $K(\mathbb{Z}/2^{[3]}, 1)$  by means of the covering  $K(\mathbf{H}_c, 1) \rightarrow K(\mathbb{Z}/2^{[3]}, 1)$ .

Because the classifying map  $\zeta$  is quaternionic, the submanifold  $\tilde{L}^{n-4k-4} \subset L^{n-4k}$  is co-oriented and represents the homological Euler class of the  $SO(4)$ -

bundle  $\lambda^*(\psi_{\mathbf{Q}})$ , and moreover for the corresponding  $O(4)$ -bundles  $\bar{\zeta}^*(\psi_{\mathbf{H}_c}) = \bar{\lambda}^*(\psi_{\mathbf{Q}}^!)$ , where:

–  $\psi_{\mathbf{Q}}$  is the universal  $SO(4)$ -vector bundle over the classifying space  $K(\mathbf{Q}, 1)$ . This bundle is given by the quaternionic-conjugated representation with respect to the representation (27)–(29). The bundle  $\psi_{\mathbf{Q}}$ , as a  $O(4)$ -bundle, is defined by the formula:  $\psi_{\mathbf{Q}} = i_a^*(\psi_{[3]})$ ,  $\psi_{\mathbf{Q}}^! = i_{\mathbf{I}_a, \mathbf{Q}}^*(\psi_{\mathbf{Q}})$ .

–  $\psi_{\mathbf{H}_c}$  is the universal  $O(4)$ -bundle over  $K(\mathbf{H}_c, 1)$  ( $\mathbf{H}_c \cong \mathbf{D} \times \mathbf{D}$ ).

–  $\bar{\lambda} : \bar{L}^{n-4k} \rightarrow K(\mathbf{I}_a, 1)$  is the 2-sheeted covering over the classifying mapping  $\lambda : L^{n-4k} \rightarrow K(\mathbf{Q}, 1)$ , induced by the 2-sheeted covering  $K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{Q}, 1)$  over the target space of the map  $\lambda$ .

For the universal  $SO(4)$ -bundle  $\psi_{\mathbf{Q}}^!$  the following formula is satisfied:

$$\psi_{\mathbf{Q}}^! = \psi_+ \oplus \psi_-, \quad (72)$$

where the bundle  $\psi_+$  admits a lift  $\psi_+^U$  to a complex  $U(1)$ -bundle, the bundle  $\psi_-$  is a  $SO(2)$ -bundle, obtained from  $\psi_+^U$  by means of the complex conjugation and forgetting the complex structure.

The proof of (72) follows from the formulas (27)–(29). This formulas correspond to Lemma 16:  $\psi_- = T_{\mathbf{Q}}^*(\psi_+)$ .

The bundles  $\psi_+$ ,  $\psi_-$  satisfy the equation:  $e(\psi_+) = -e(\psi_-)$ , and the Euler class  $e(\psi_+)$  of the bundle  $\psi_+$  is equal to the generator  $t \in H^2(K(\mathbf{I}_a, 1); \mathbb{Z})$  in the standard basis, the Euler class  $e(\psi_-)$  of the bundle  $\psi_-$  is equal to  $-t$  and is opposite to the generator  $t$  of the standard basis.

Let us denote by  $m \in H^{4k}(N^{n-2k}; \mathbb{Z})$  the cohomology class, dual to the fundamental class of the oriented submanifold  $\bar{L}^{n-4k} \subset N^{n-2k}$  in the oriented manifold  $N^{n-2k}$ . Let us denote by  $e_g \in H^{4k}(N^{n-2k}; \mathbb{Z})$  the Euler class of the immersion  $g$  (this is the top class of the normal bundle  $\nu_g$ ). By the Herbert theorem for the immersion  $g : N^{n-2k} \looparrowright \mathbb{R}^n$  with the self-intersection manifold  $L^{n-4k}$  (see [E-G], Theorem 1.1 the case  $r = 1$ , the coefficients is  $\mathbb{Z}$ ) the following formula are satisfied:

$$e_g + m = 0. \quad (73)$$

Let us denote by  $\tilde{m} \in H^{4k-4}(N^{n-2k}; \mathbb{Z})$  the cohomology class, dual to the fundamental class of the oriented submanifold  $\bar{\tilde{L}}^{n-4k-4} \subset \tilde{N}^{n-2k-2} \subset N^{n-2k}$  in the oriented manifold  $N^{n-2k}$ . Let us denote by  $e_{\tilde{g}} \in H^{4k-4}(N^{n-2k}; \mathbb{Z})$  the cohomology class, dual to the image of the homology Euler class of the immersion  $\tilde{g}$  by the inclusion  $\tilde{N}^{n-2k-2} \subset N^{n-2k}$ . By the Herbert theorem for the immersion  $\tilde{g} : N^{n-2k} \looparrowright \mathbb{R}^n$  with the self-intersection manifold  $\tilde{L}^{n-4k}$  (see. [E-G], Theorem 1.1 the case  $r = 1$ , the coefficients is  $\mathbb{Z}$ ) the following formula are satisfied:

$$e_{\tilde{g}} + \tilde{m} = 0. \quad (74)$$

Because  $\bar{\lambda} = \mu_a$ , we may use the equation:  $\bar{\lambda}^*(\psi_{\mathbf{Q}}^!) = \mu_a^*(\psi_+) \oplus \mu_a^*(\psi_-)$ . The following equation are satisfied:  $\tilde{m} = me(\mu_a^*(\psi_+))e(\mu_a^*(\psi_-))$ , where the right side is the product of the three cohomology classes:  $m$  and the two Euler classes of the corresponding bundles. The following equation are satisfied:  $e_{\tilde{g}} = e_g e^2(\mu_a^*(\psi_+))$ . The equation (74) can be rewritten in the following form:

$$e_g e^2(\mu_a^*(\psi_+)) + me(\mu_a^*(\psi_+))e(\mu_a^*(\psi_-)) = 0. \quad (75)$$

Then we may take into account (73) and the equation  $e(\mu_a^*(\psi_-)) = -e(\mu_a^*(\psi_+))$ . Let us rewrite the previous formula as follows:

$$2e_g e^2(\mu_a^*(\psi_+)) = 0. \quad (76)$$

Because of the equation  $e_g = e(\mu_a^*(\psi_+))^k$ , we obtain:

$$2e^{k+2}(\mu_a^*(\psi_+)) = 0. \quad (77)$$

Let us recall that  $\dim(L) = n - 4k = n_{\sigma} \geq 7$  and  $\dim(\tilde{L}) = n_{\sigma} - 4 \geq 3$ . The formula for the Hopf invariant for  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$  using (32) is the following:

$$h_k^{\mathbf{D}}((g, \eta, \Psi)) = \langle e^{k+2}(\mu_a^*(\psi_+))\mu_a^*(\tau_{n-4k-4}); [N^{n-2k}] \rangle \pmod{2}, \quad (78)$$

where  $\tau_{n-4k-4} \in H^{n-4k-4}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$  is the generic class modulo 4, the cohomology class  $e(\mu_a^*(\psi_+))$  is modulo 4, and the fundamental class  $[N^{n-2k}]$  of the oriented manifold  $N^{n-2k}$  is modulo 4. The condition  $h_k^{\mathbf{D}}((g, \eta, \Psi)) = 1$  implies the following condition: the cohomology class  $e^{k+2}(\mu_a^*(\psi_+))$  is of order 4. This contradicts the formula (77). Therefore,  $h_k^{sf}(f, \kappa, \Xi) = h_k^{\mathbf{D}}((g, \eta, \Psi)) = 0$  and the theorem in the particular case is proved.

Let us prove the theorem in the general case. Let us consider the pair of mappings  $(\mu_a, \lambda)$ , where  $\mu_a : N_a^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$ ,  $N_a^{n-2k} \subset N^{n-2k}$ ,  $\lambda = \lambda_{\mathbf{Q}} \cup \lambda_{\mathbf{H}_{b \times b}} : L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_{b \times b}}^{n-4k} \rightarrow K(\mathbf{Q}, 1) \cup K(\mathbf{H}_{b \times b}, 1)$ , where  $L_a^{n-4k} = L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_{b \times b}}^{n-4k}$ ,  $L_a^{n-4k} \subset L^{n-4k}$ , these two mappings determine the quaternionic structure of the  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$  in the sense of Definition 19.

Let us consider the manifold  $\bar{L}_a^{n-4k} = \bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_{b \times b}}^{n-4k}$ , defined by the formula (37). The manifold  $\bar{L}_a^{n-4k}$  is the canonical 2-sheeted covering over the manifold  $L_a^{n-4k}$ .

The formula (73) is valid, and additionally the cohomology class  $m$  (this class is dual to the fundamental class  $[\bar{L}_a]$  of the submanifold  $\bar{L}_a^{n-4k} \subset N_a^{n-2k}$ ) decomposes into the following sum:

$$m = m_{\mathbf{Q}} + m_{\mathbf{H}_{b \times b}}, \quad (79)$$

corresponding to the type of the components  $L_{\mathbf{Q}}^{n-4k}$ ,  $L_{\mathbf{H}_{b \times b}}^{n-4k}$  of the self-intersection manifold (see the formula (40)).

Let us consider the submanifold  $\tilde{N}_a^{n-2k-2} \subset N_a^{n-2k}$ , representing the Euler class of the bundle  $\mu_a^*(\psi^+)$ . The following immersion  $\tilde{g}_a : \tilde{N}_a^{n-2k-2} \looparrowright \mathbb{R}^n$  is well defined by the restriction of the immersion  $g_a$  to the submanifold  $\tilde{N}_a^{n-2k-2} \subset N_a^{n-2k}$ . Let us denote by  $\tilde{L}_a^{n-4k-4}$  the self-intersection manifold of the immersion  $\tilde{g}_a$  (compare with the corresponding definition of the previous step).

The inclusion  $\tilde{L}_a^{n-2k-4} \subset L_a^{n-2k}$  is well defined. In particular, the manifold  $\tilde{L}_a^{n-4k-4}$  is represented by the union of the following two components:  $\tilde{L}_a^{n-4k-4} = \tilde{L}_{\mathbf{Q}}^{n-4k-4} \cup \tilde{L}_{\mathbf{H}_{b \times b}}^{n-4k-4}$ .

**Lemma 32.** *The co-oriented submanifold  $\tilde{L}_{\mathbf{Q}}^{n-2k-4} \subset L_{\mathbf{Q}}^{n-2k}$  represents the Euler class of the  $SO(4)$ -bundle  $\lambda_{\mathbf{Q}}^*(\psi_{\mathbf{Q}})$ .*

*The submanifold  $\tilde{L}_{\mathbf{H}_{b \times b}}^{n-2k-4} \subset L_{\mathbf{H}_{b \times b}}^{n-2k}$  represents the Euler class of the  $SO(4)$ -bundle  $\lambda_{\mathbf{H}_{b \times b}}^*(\psi_{\mathbf{H}_{b \times b}})$ , where  $\psi_{\mathbf{H}_{b \times b}}$  is the universal  $SO(4)$ -bundle over the space  $K(\mathbf{H}_{b \times b}, 1)$ . The corresponding  $O(4)$ -bundle is standardly defined as the inverse image of the bundle  $\psi_{\mathbb{Z}/2^{[3]}}$  over  $K(\mathbb{Z}/2^{[3]}, 1)$  by means of the inclusion  $K(\mathbf{H}_{b \times b}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$ .*

### Proof of Lemma 32

The proof follows from the arguments above in the proof of commutativity of the left squares of the diagrams (9) and (26).

The bundle  $\psi_{\mathbf{H}_{b \times b}}$  is isomorphic to the Whitney sum of the two 2-bundles:  $\psi_{\mathbf{H}_{b \times b}} = p_{\mathbf{H}_{b \times b}, \mathbf{I}_a}^*(\psi_{\mathbf{I}_a}) \oplus p_{\mathbf{H}_{b \times b}, \mathbf{I}_a}^*(\psi_{\mathbf{I}_a}) \otimes l_{\mathbb{Z}/2}$ , where  $p_{\mathbf{H}_{b \times b}, \mathbf{I}_a}^*(\psi_{\mathbf{I}_a})$  is the 2-dimensional bundle, defined as the pull-back of the canonical 2-dimensional bundle  $\psi^+$  over  $K(\mathbf{I}_a, 1)$  by means of the natural mapping  $p_{\mathbf{H}_{b \times b}, \mathbf{I}_a} : K(\mathbf{H}_{b \times b}, 1) \rightarrow K(\mathbf{I}_a, 1)$ , induced by the homomorphism  $p_{\mathbf{H}_{b \times b}, \mathbf{I}_a} : \mathbf{H}_{b \times b} \rightarrow \mathbf{I}_a$ ,  $l_{\mathbb{Z}/2}$  is a line bundle, defined as the inverse image of the canonical line bundle over  $K(\mathbb{Z}/2, 1)$  by means of the projection  $K(\mathbf{H}_{b \times b}, 1) \rightarrow K(\mathbb{Z}/2, 1)$ , this projection corresponds to the epimorphism  $\mathbf{H}_{b \times b} \rightarrow \mathbb{Z}/2$  with the kernel  $\mathbf{I}_a \subset \mathbf{H}_{b \times b}$ .

By analogous arguments the class  $\tilde{m}$  is well defined as in the formula (74), moreover, the following formula is satisfied:

$$\tilde{m} = \tilde{m}_{\mathbf{Q}} + \tilde{m}_{\mathbf{H}_{b \times b}}, \quad (80)$$

where the terms in the right side of the formula are defined as the cohomology classes, dual to the fundamental classes  $[\tilde{L}_{\mathbf{Q}}]$ ,  $[\tilde{L}_{\mathbf{H}_{b \times b}}]$  of the canonical coverings over the corresponding component.

The formula relating  $m_{\mathbf{Q}}$  and  $\tilde{m}_{\mathbf{Q}}$  is the following:  $\tilde{m}_{\mathbf{Q}} = m_{\mathbf{Q}} e(\mu_a^*(\psi_+)) e(\mu_a^*(\psi_-))$ . The formula relating  $m_{\mathbf{H}_{b \times b}}$  and  $\tilde{m}_{\mathbf{H}_{b \times b}}$  is the following:  $\tilde{m}_{\mathbf{H}_{b \times b}} = m_{\mathbf{H}_{b \times b}} e^2(\mu_a^*(\psi_+))$ . To prove the last equation we use the following fact: the bundle  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}}^*(\psi_{\mathbf{H}_{b \times b}})$ , where the mapping  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}} : K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{H}_{b \times b}, 1)$  corresponds to the index 2 subgroup  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}} : \mathbf{I}_a \subset \mathbf{H}_{b \times b}$ , is isomorphic to the bundle  $\psi^+ \oplus \psi^+$ .

The analog of the formula (75) is the following:

$$e_g e^2(\mu_a^*(\psi_+)) - m_{\mathbf{Q}} e^2(\mu_a^*(\psi_+)) + m_{\mathbf{H}_{b \times b}} e^2(\mu_a^*(\psi_+)) = 0. \quad (81)$$

Let us multiply both sides of the formula (79) by the cohomology class  $e^2(\mu_a^*(\psi_+))$  and take the sum with the opposite sign with (81), we get:

$$2m_{\mathbf{Q}} e^2(\mu_a^*(\psi_+)) = 0. \quad (82)$$

This is an analog of the formula (76).

Let us prove that the Hopf invariant of the  $\mathbf{D}$ -framed immersion  $(g, \eta, \Psi)$  is trivial. By Corollary 22 the Hopf invariant is given by the formula (40). Let us prove that the each term in this formula is equal to zero. The first term  $h_\lambda(L_{\mathbf{Q}})$ , according to (35), is calculated as the reduction modulo 2 of the following characteristic number modulo 4:

$$h_\lambda(L_{\mathbf{Q}}) = \langle m_{\mathbf{Q}} \mu_a^*(x); [N_a^{n-2k}] \rangle,$$

where  $x \in H^{n-4k}(\mathbf{I}_a; \mathbb{Z}/4)$  is the generator. Analogously, the second term  $h_\lambda(L_{\mathbf{H}_{b \times b}})$  is the reduction modulo 2 of the following number modulo 4:

$$h_\lambda(L_{\mathbf{H}_{b \times b}}) = \langle m_{\mathbf{H}_{b \times b}} \mu_a^*(x); [N_a^{n-2k}] \rangle.$$

Note that  $x = \tau^2 y$ , where  $\tau \in H^2(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$ ,  $y \in H^{n-4k-4}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$  are the generators. We have  $\mu_a(\tau) = e(\mu_a(\psi^+))$ , because  $\tau$  is the Euler class of the bundle  $\psi_+$ . Therefore, from (82) we get

$$h_\lambda(L_{\mathbf{Q}}) = 0,$$

because  $m_{\mathbf{Q}} e^2(\mu_a^*(\psi_+)) = m_{\mathbf{Q}} (\mu_a^*(\tau))^2 = m_{\mathbf{Q}} \mu_a^*(x)$ .

To calculate the second term  $h_\lambda(L_{\mathbf{H}_{b \times b}})$  it is sufficient to note that

$$\langle m_{\mathbf{H}_{b \times b}} \mu_a^*(x); [N_a^{n-2k}] \rangle = \langle \mu_a^*(x); [\bar{L}_{\mathbf{H}_{b \times b}}^{n-4k}] \rangle = \langle p^*((\lambda_{\mathbf{H}_{b \times b}})^*(x')); [\bar{L}_{\mathbf{H}_{b \times b}}^{n-4k}] \rangle = 0,$$

where  $p : \bar{L}_{\mathbf{H}_{b \times b}}^{n-4k} \rightarrow L_{\mathbf{H}_{b \times b}}^{n-4k}$  – is the 2-sheeted covering, corresponding to the subgroup  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}}$ ,  $x' \in H^{n-4k}(K(\mathbf{H}_{b \times b}, 1); \mathbb{Z}/4)$  is a cohomology class, such that  $i_{\mathbf{I}_a, \mathbf{H}_{b \times b}}^*(x') = x$ ,  $[\bar{L}_{\mathbf{H}_{b \times b}}^{n-4k}]$  is the fundamental class of the total manifold of the canonical 2-sheeted covering  $p$ .

Theorem 12 is proved.

**Remark 33.** A straightforward generalization of Theorem 12 for mappings with singularities  $c : S^{n-2k}/\mathbf{i} \rightarrow \mathbb{R}^n$ , which admits a (relative) quaternionic structure in the sense of Definition 29 is not possible.

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